61A Lecture 20

Friday, October 14

Tree Recursion

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fib(n): 0, 1, 1, 2, 3, 5, 8, 13, 21,

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A Tree-Recursive Process

The computational process of fib evolves into a tree structure

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```
fib(6)
```

A Tree-Recursive Process

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fib(6)

fib(4)

A Tree-Recursive Process

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```
                fib(6)
              /        \
        fib(4)          fib(5)
```

A Tree-Recursive Process

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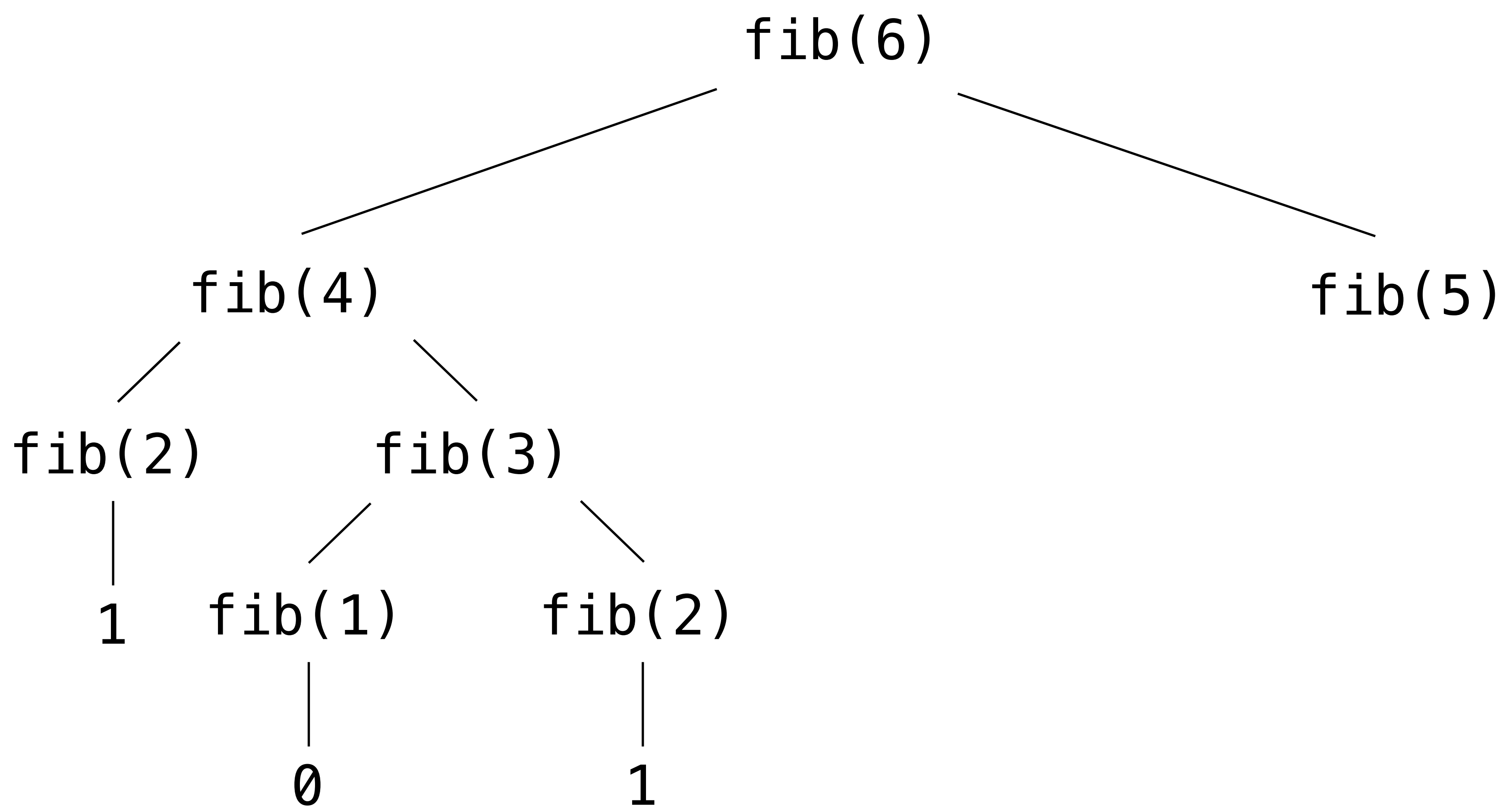

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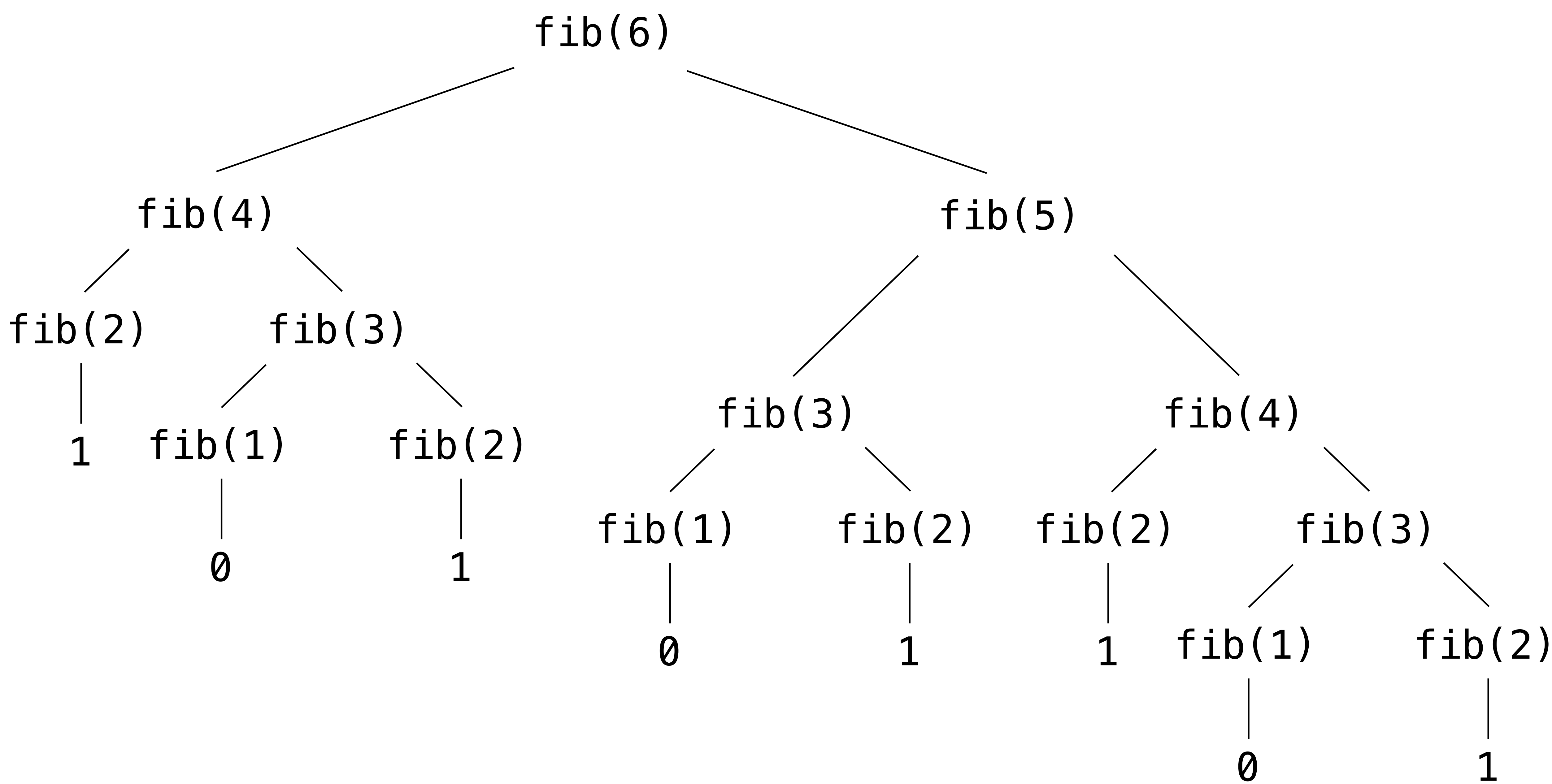

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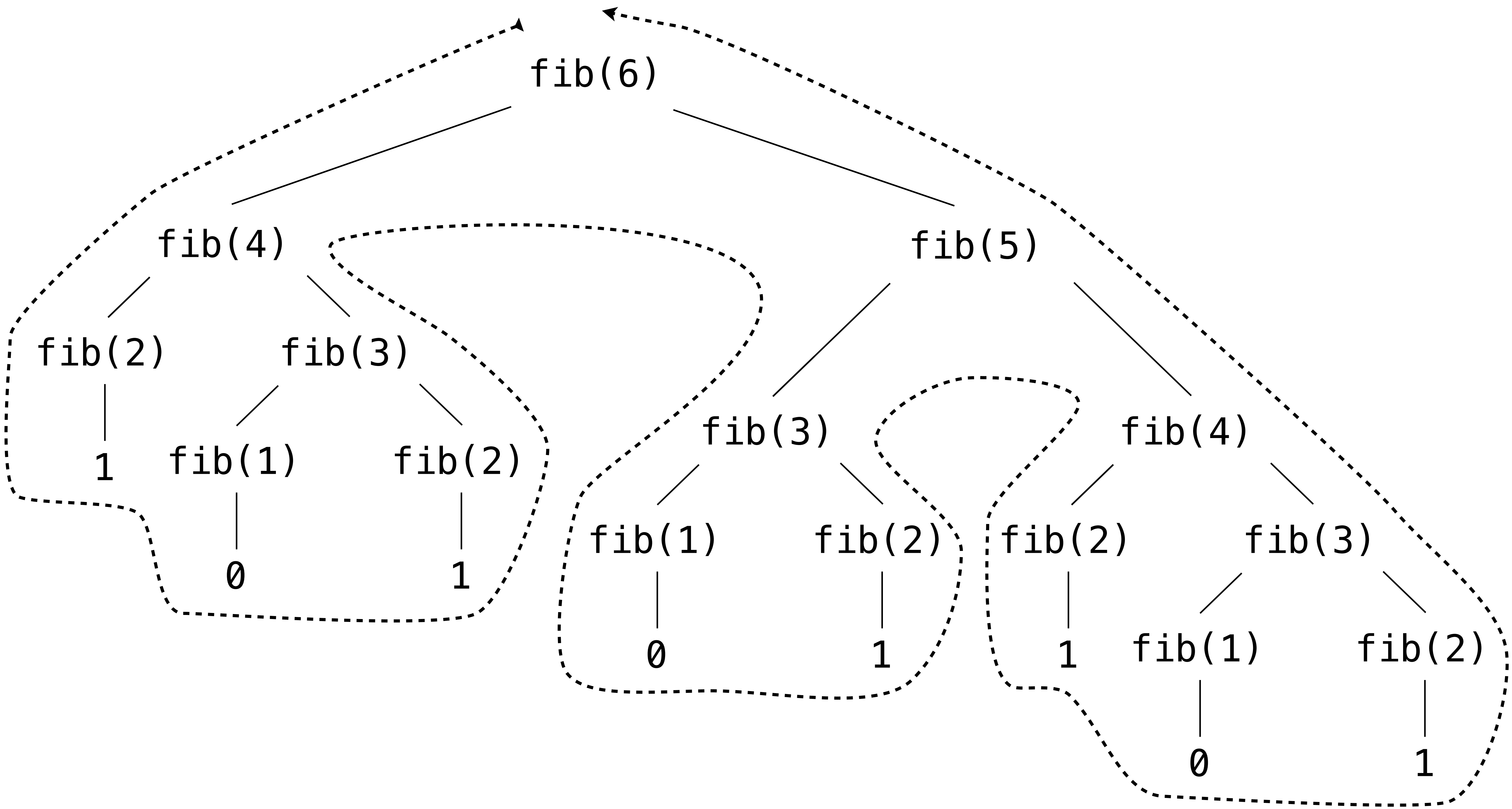

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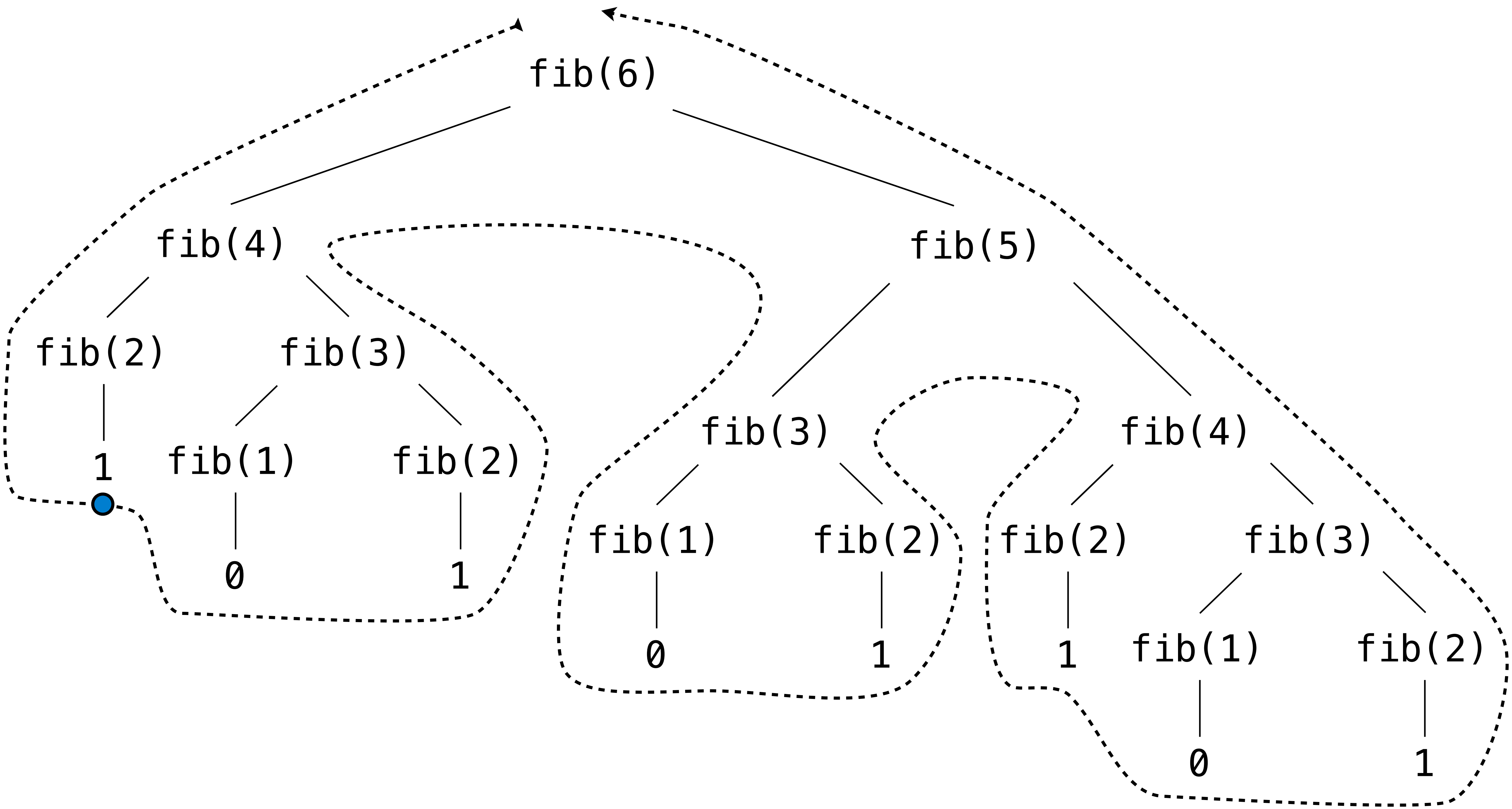

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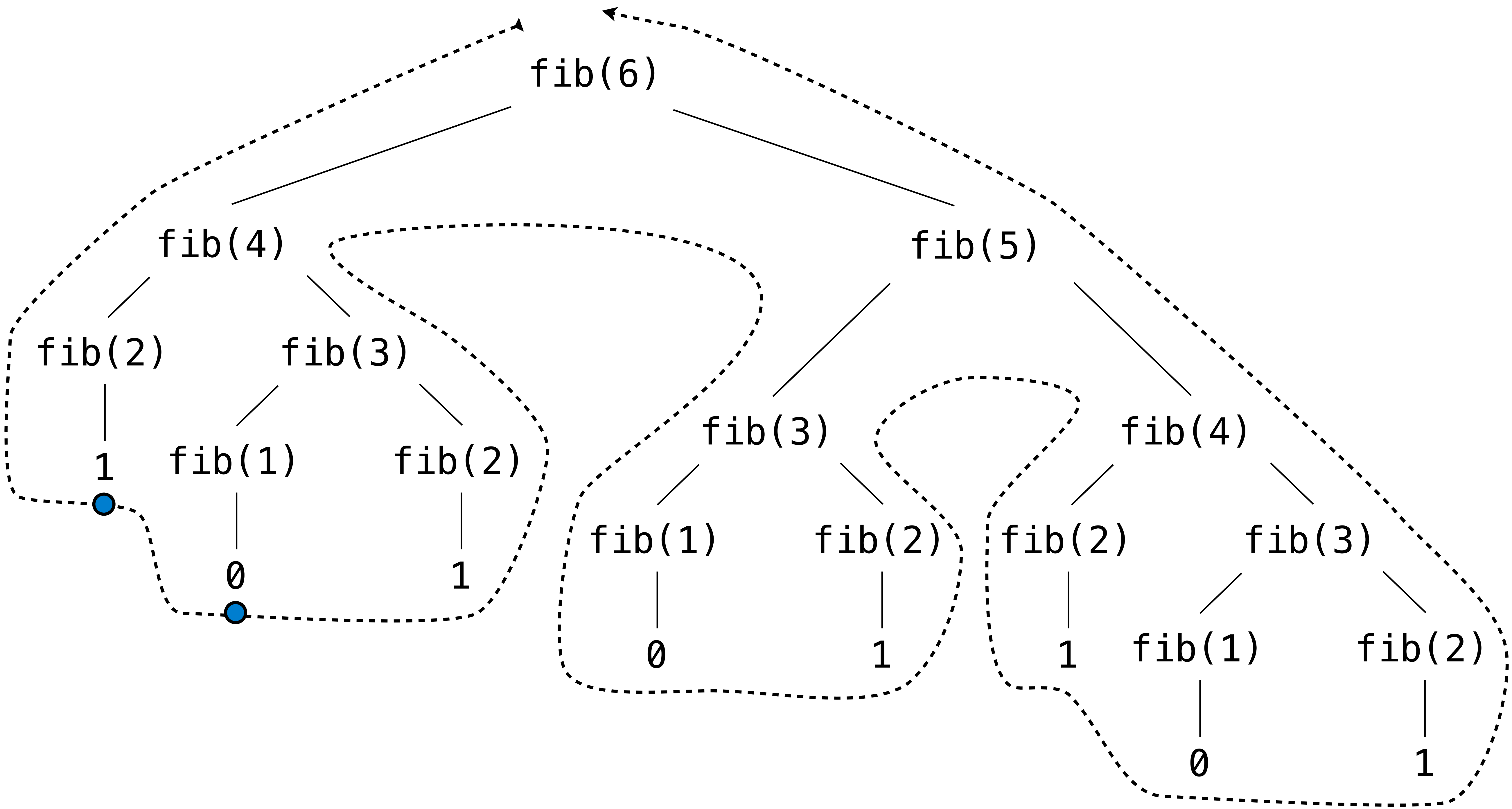

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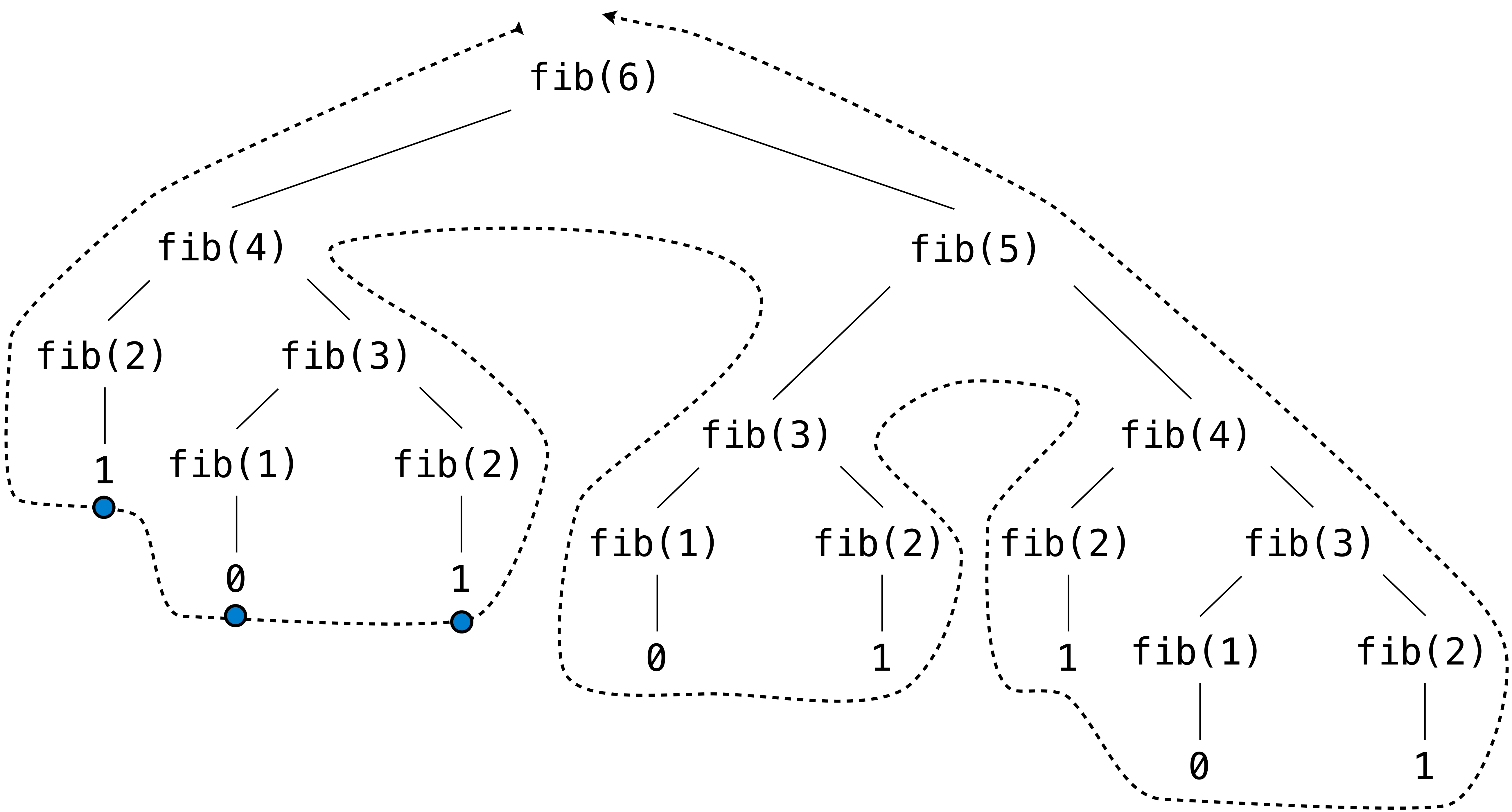

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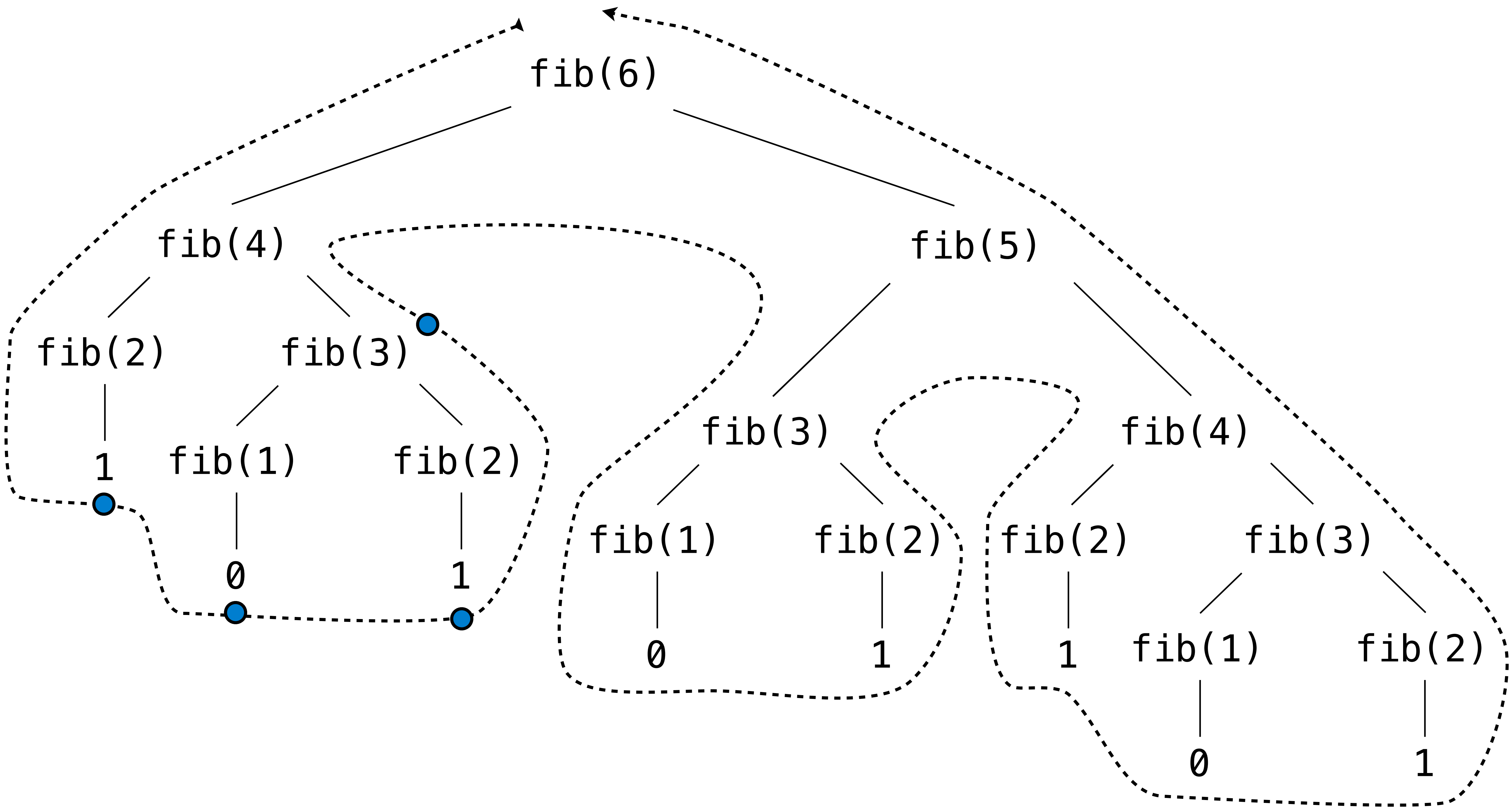

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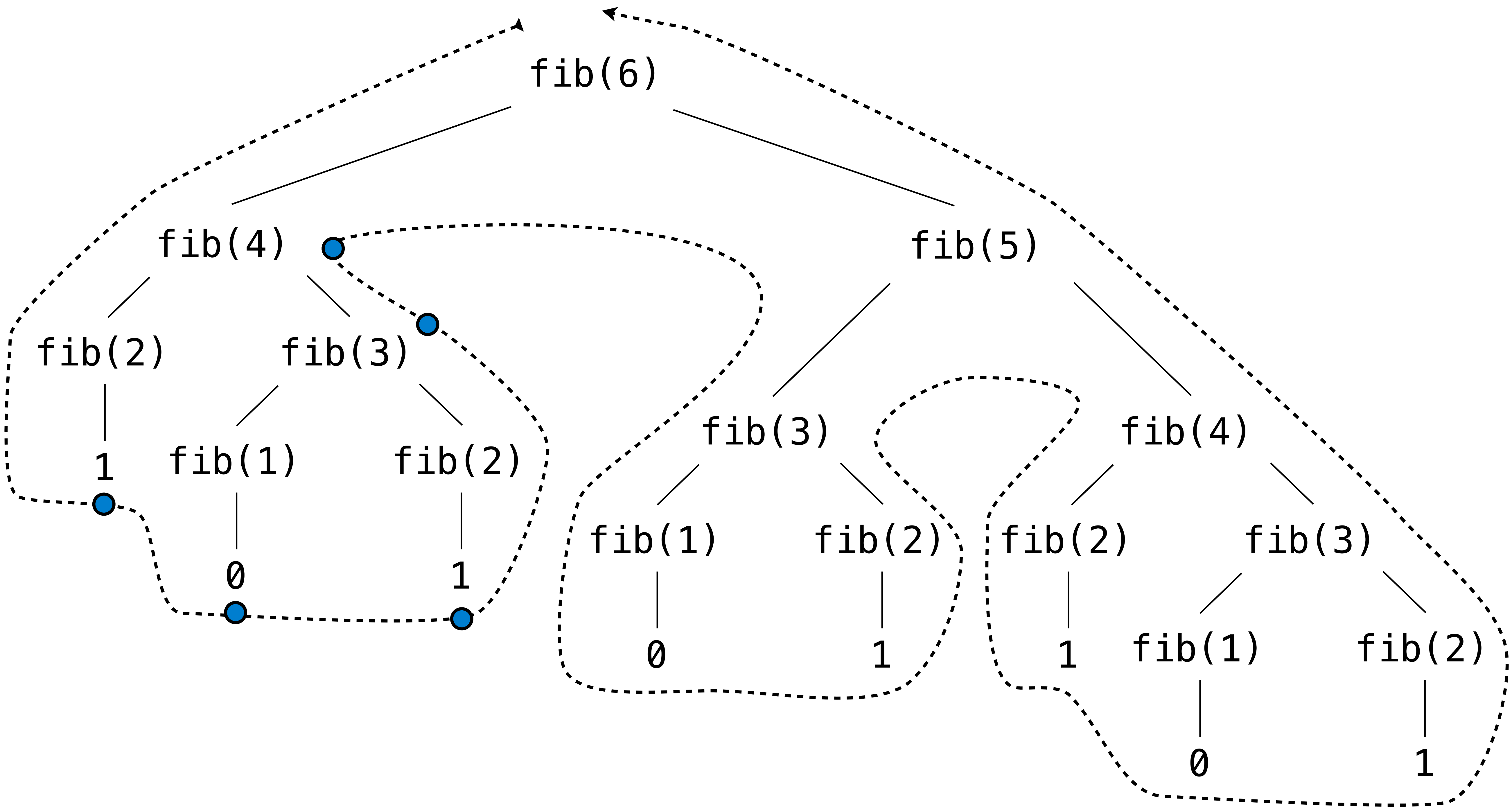

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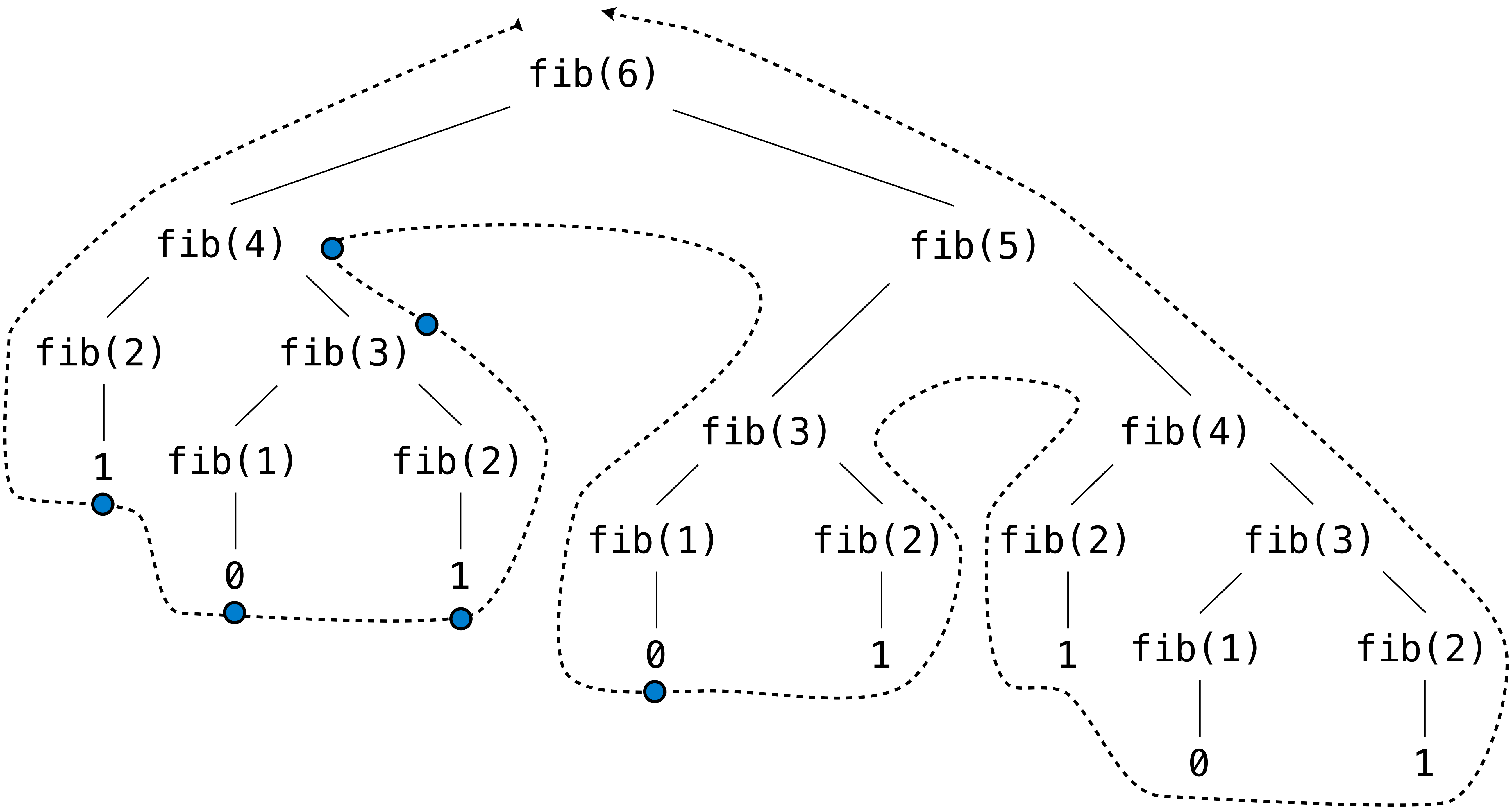

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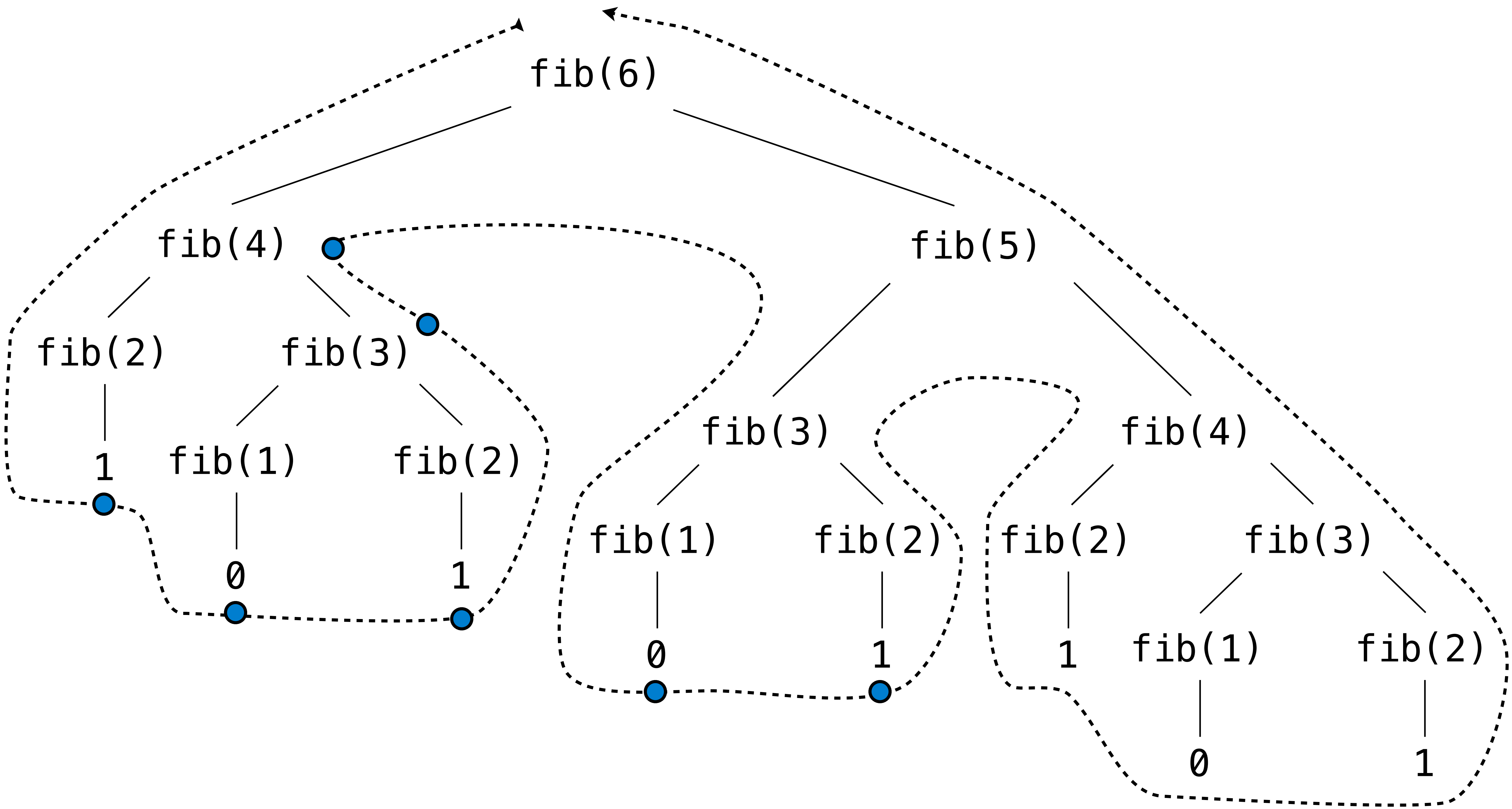

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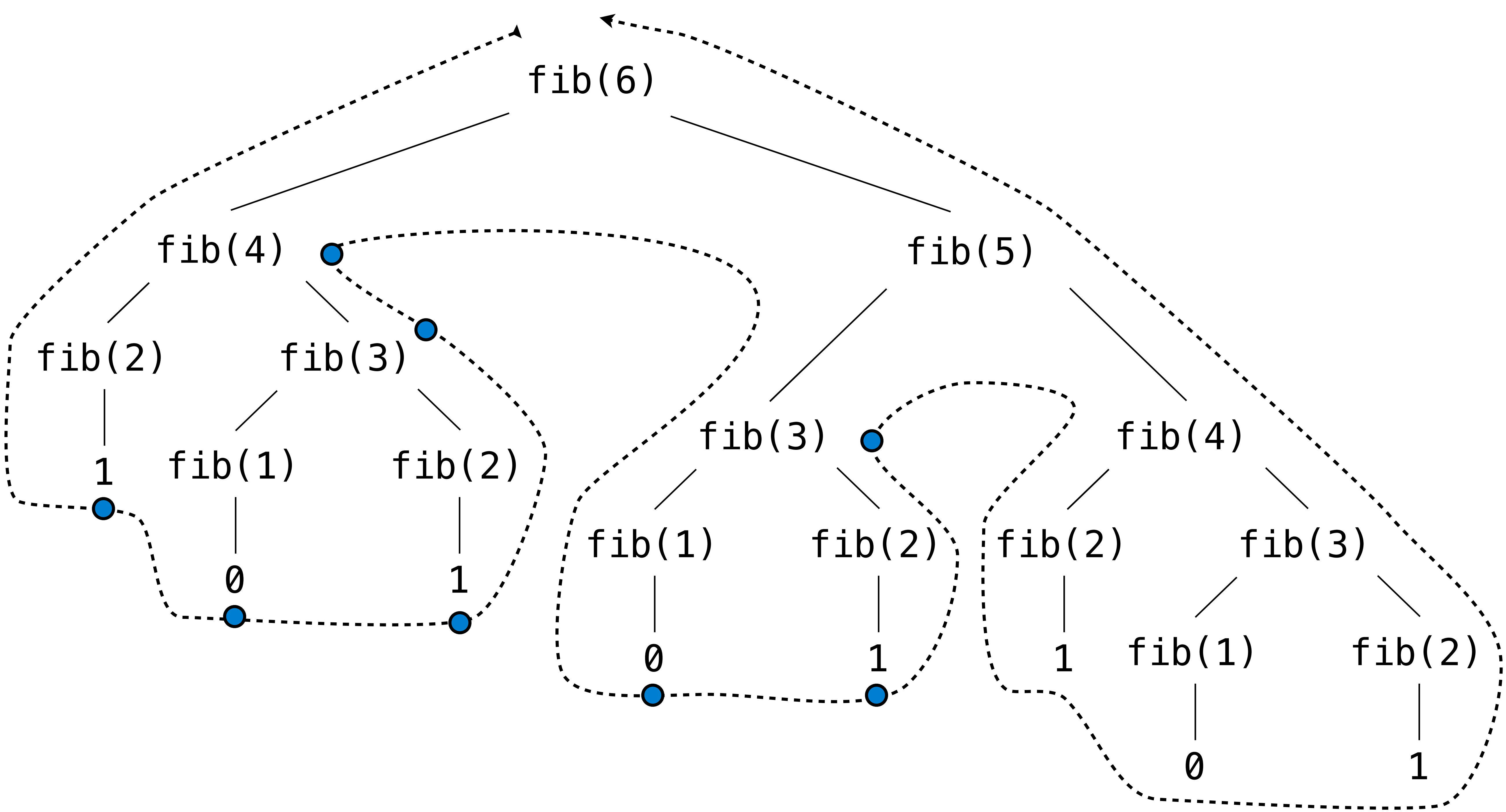

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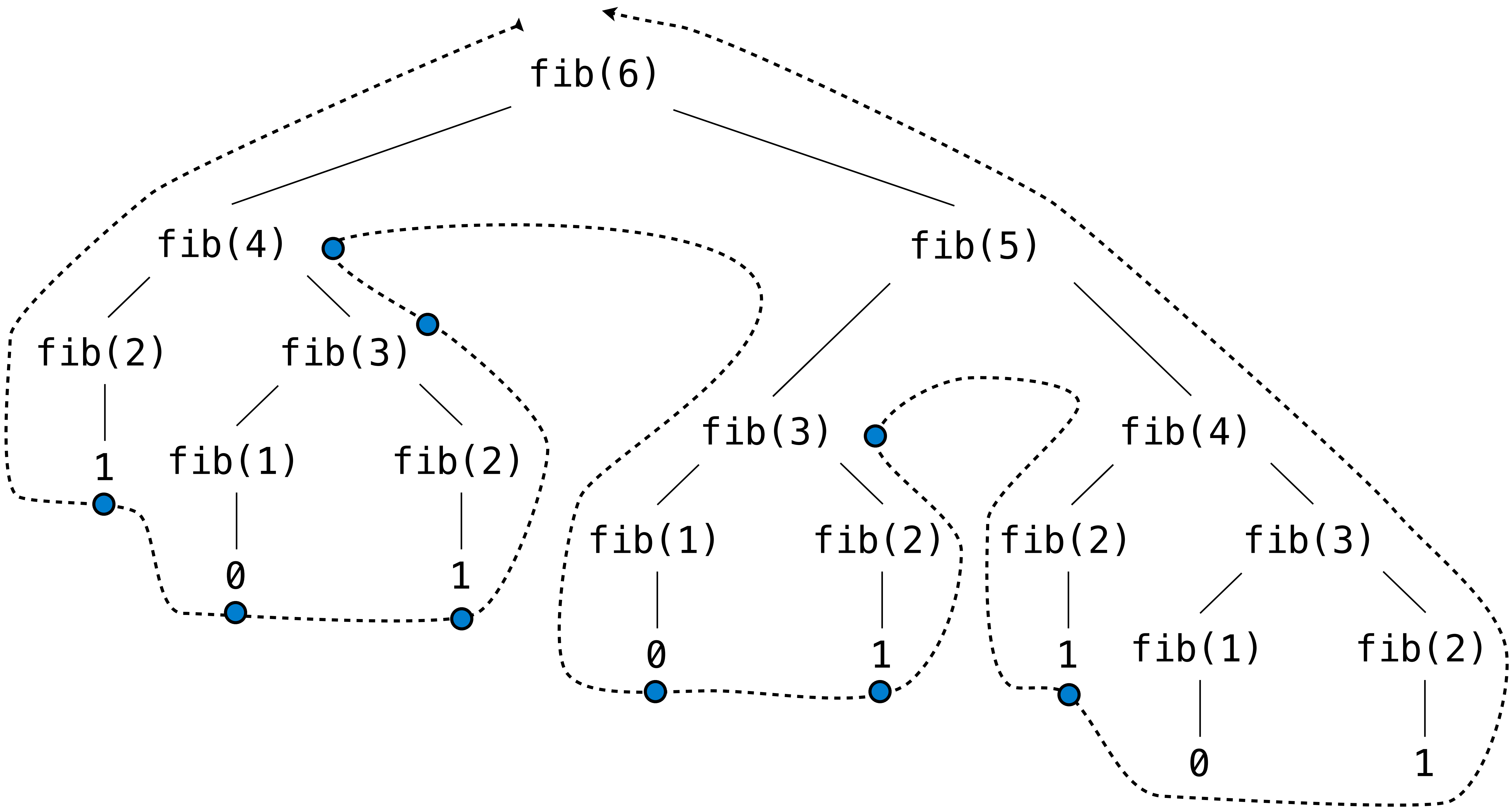

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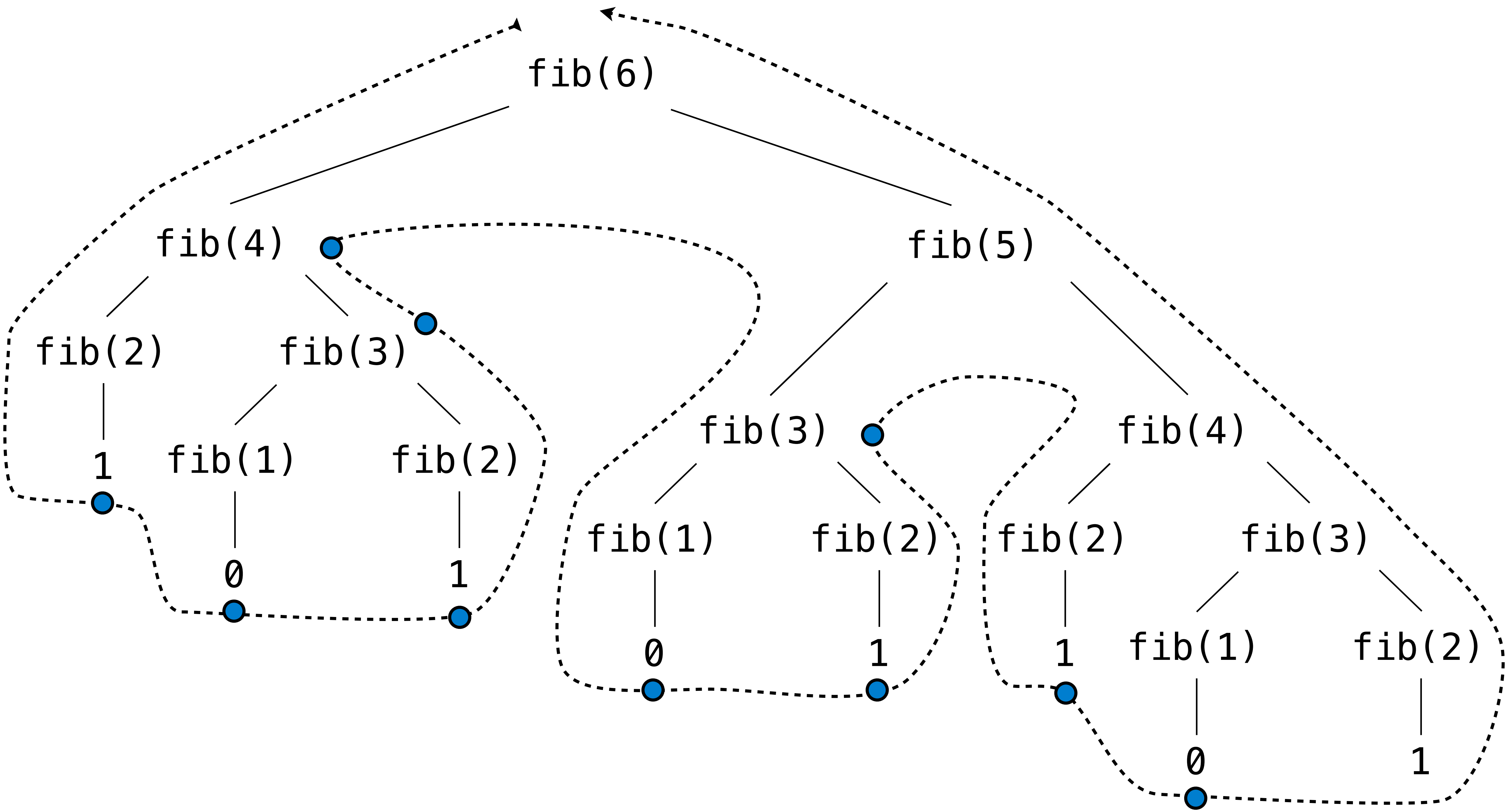

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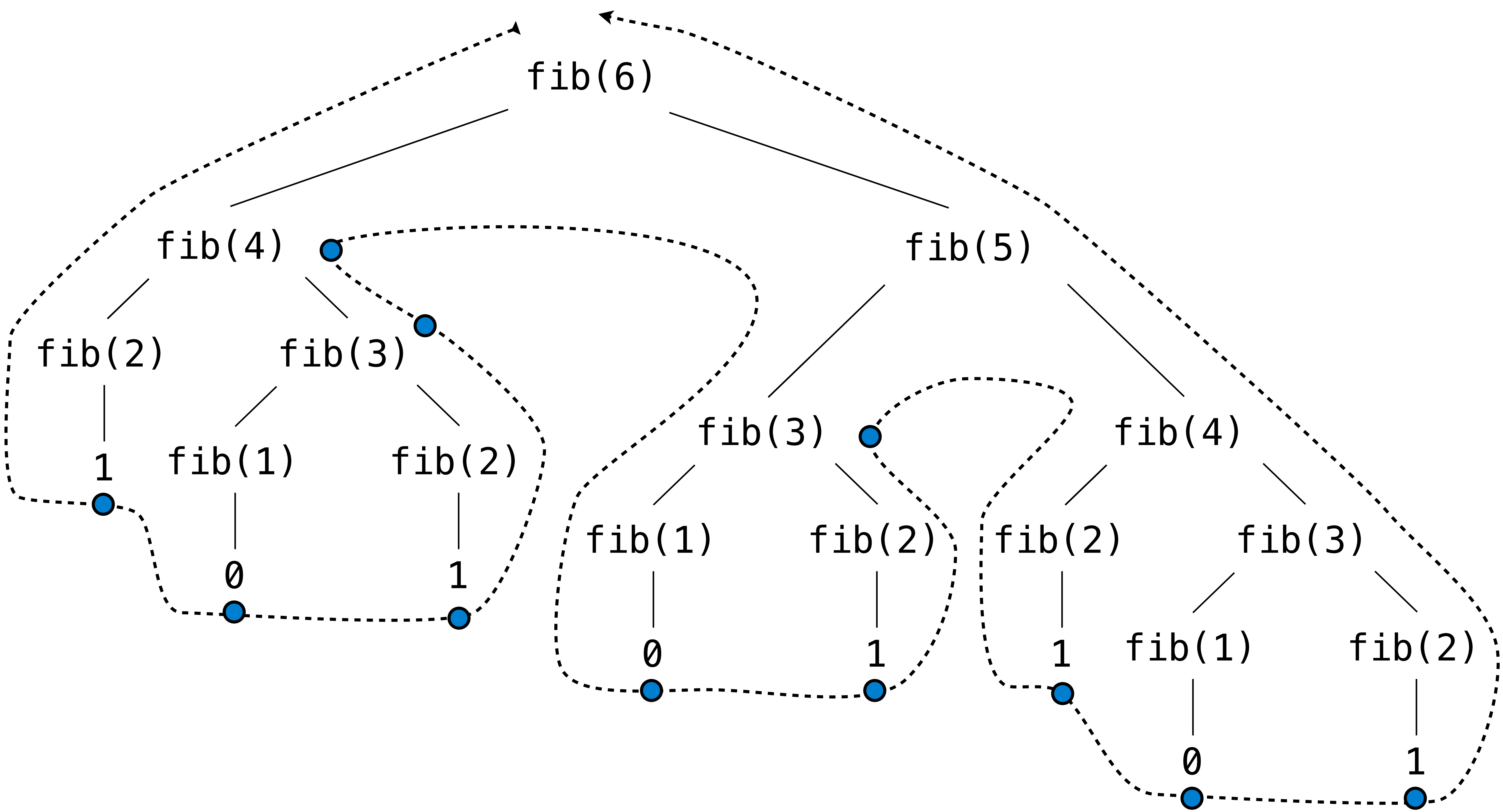

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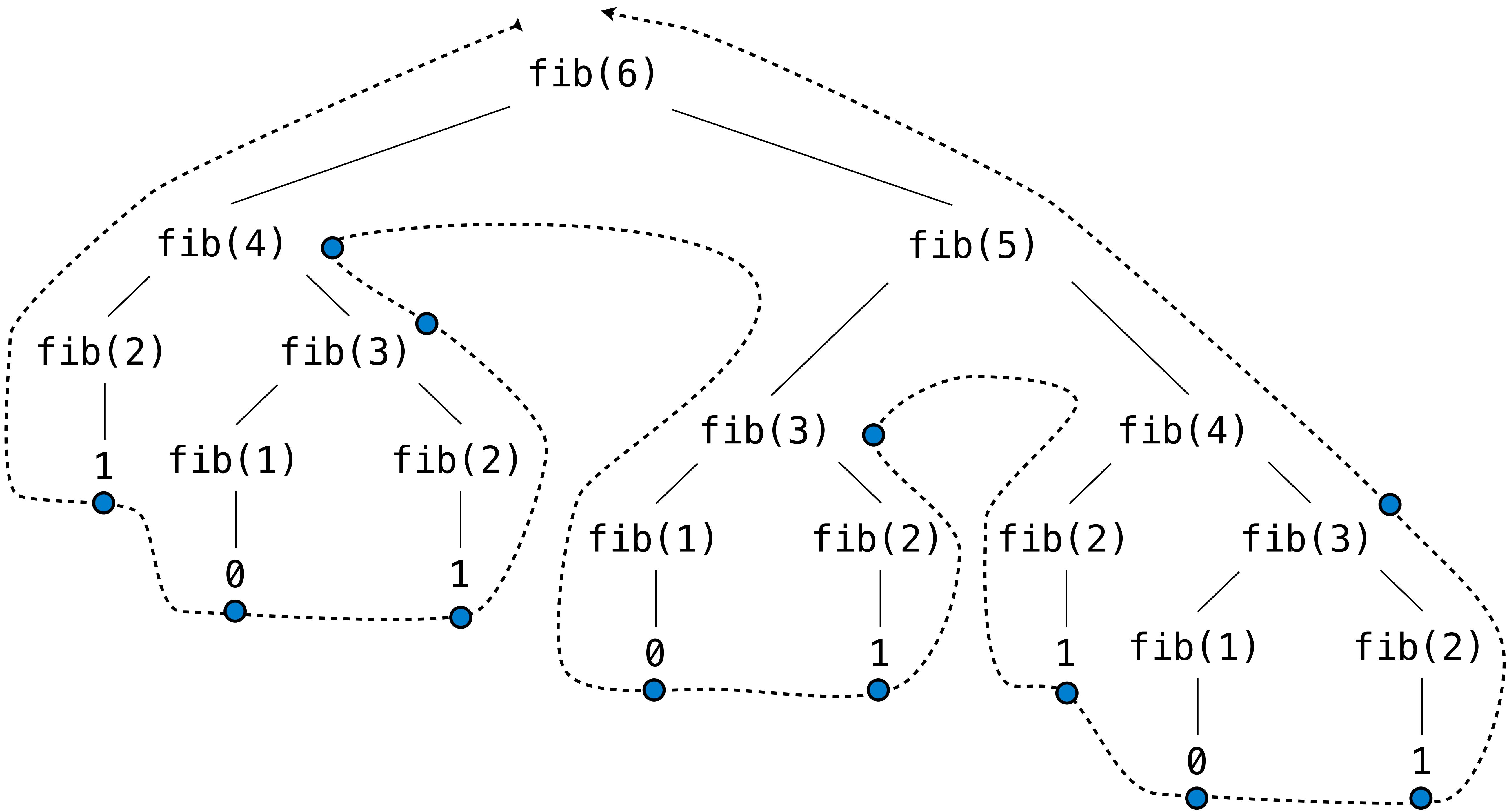

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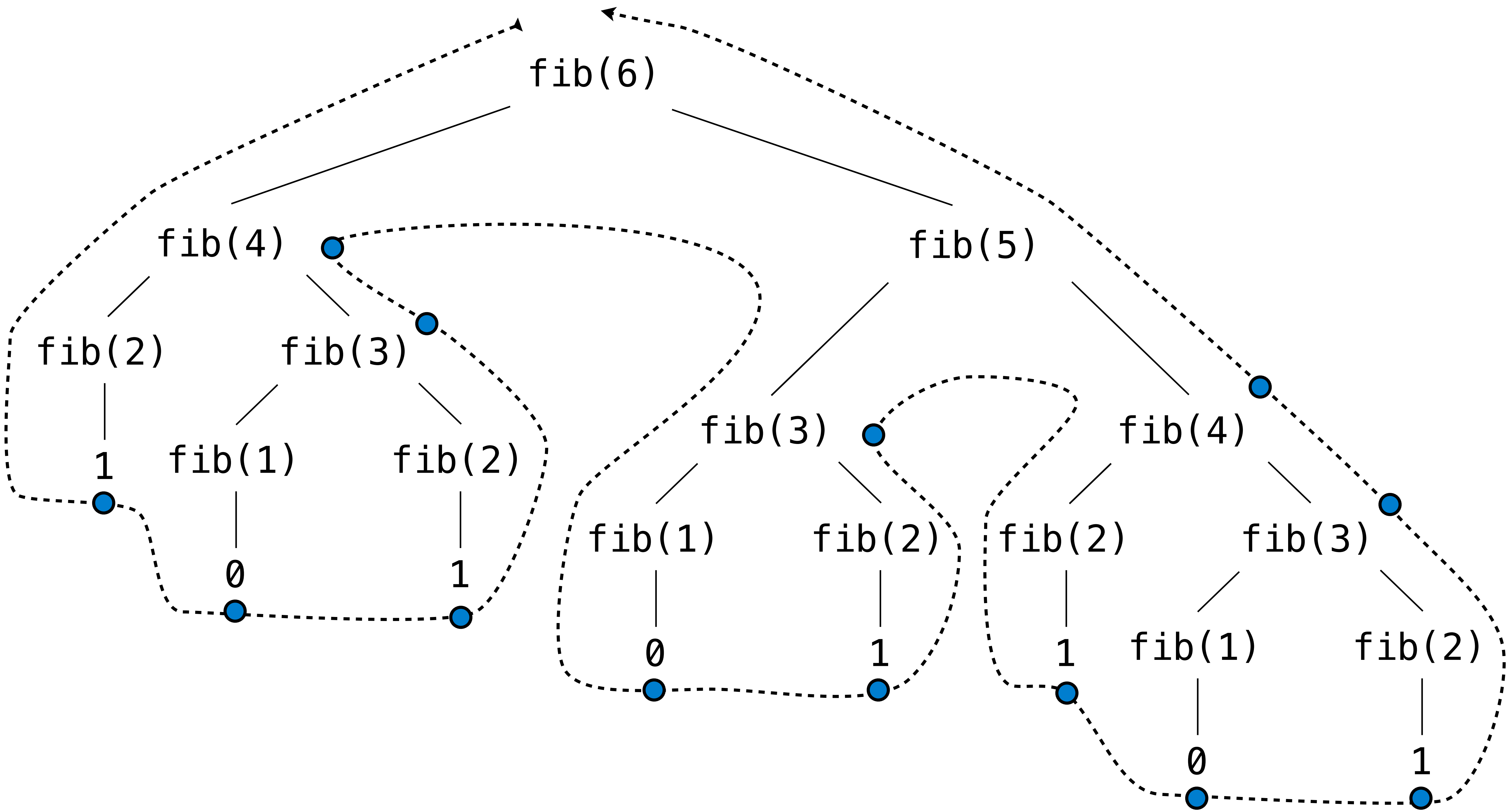

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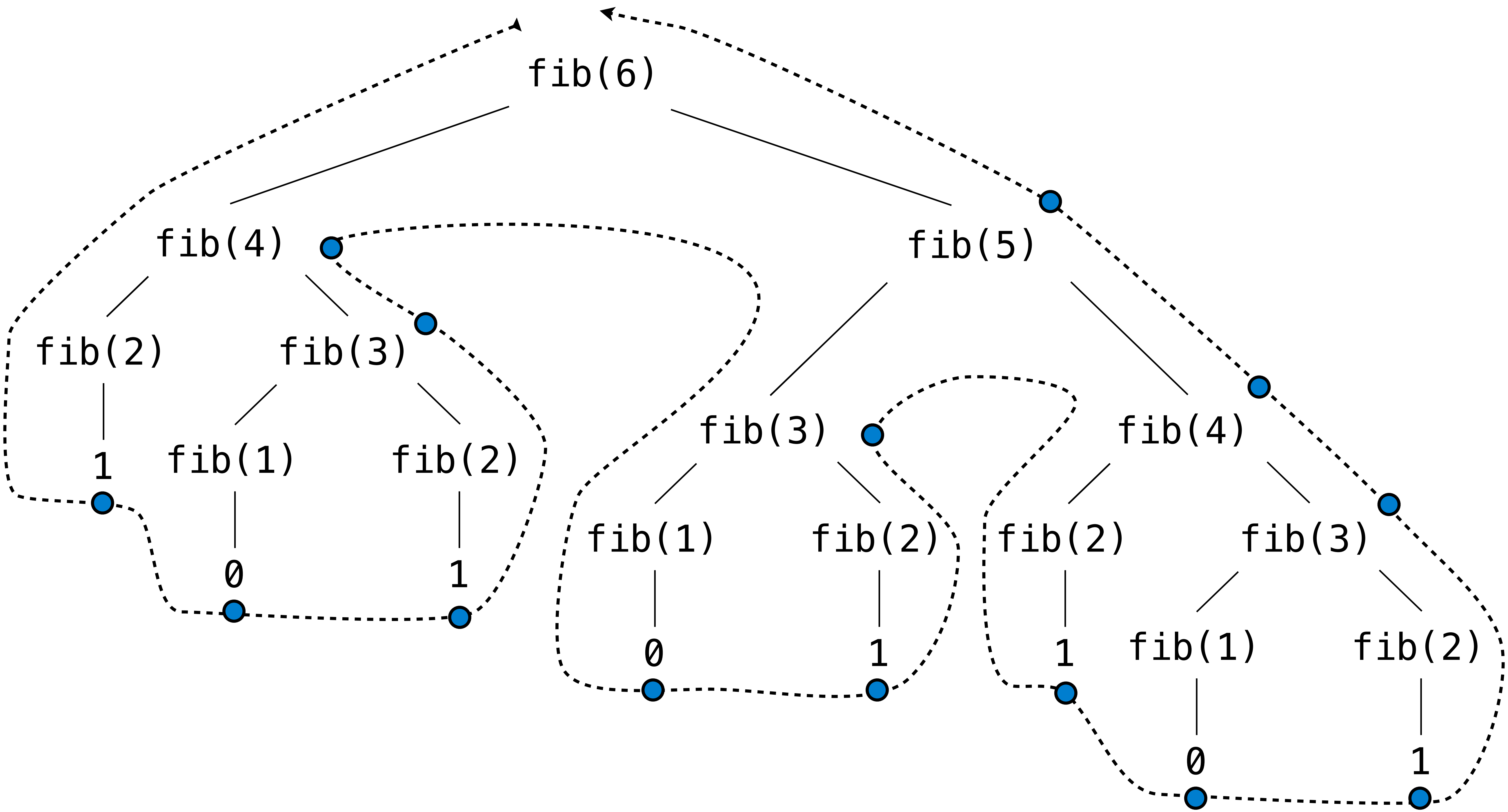

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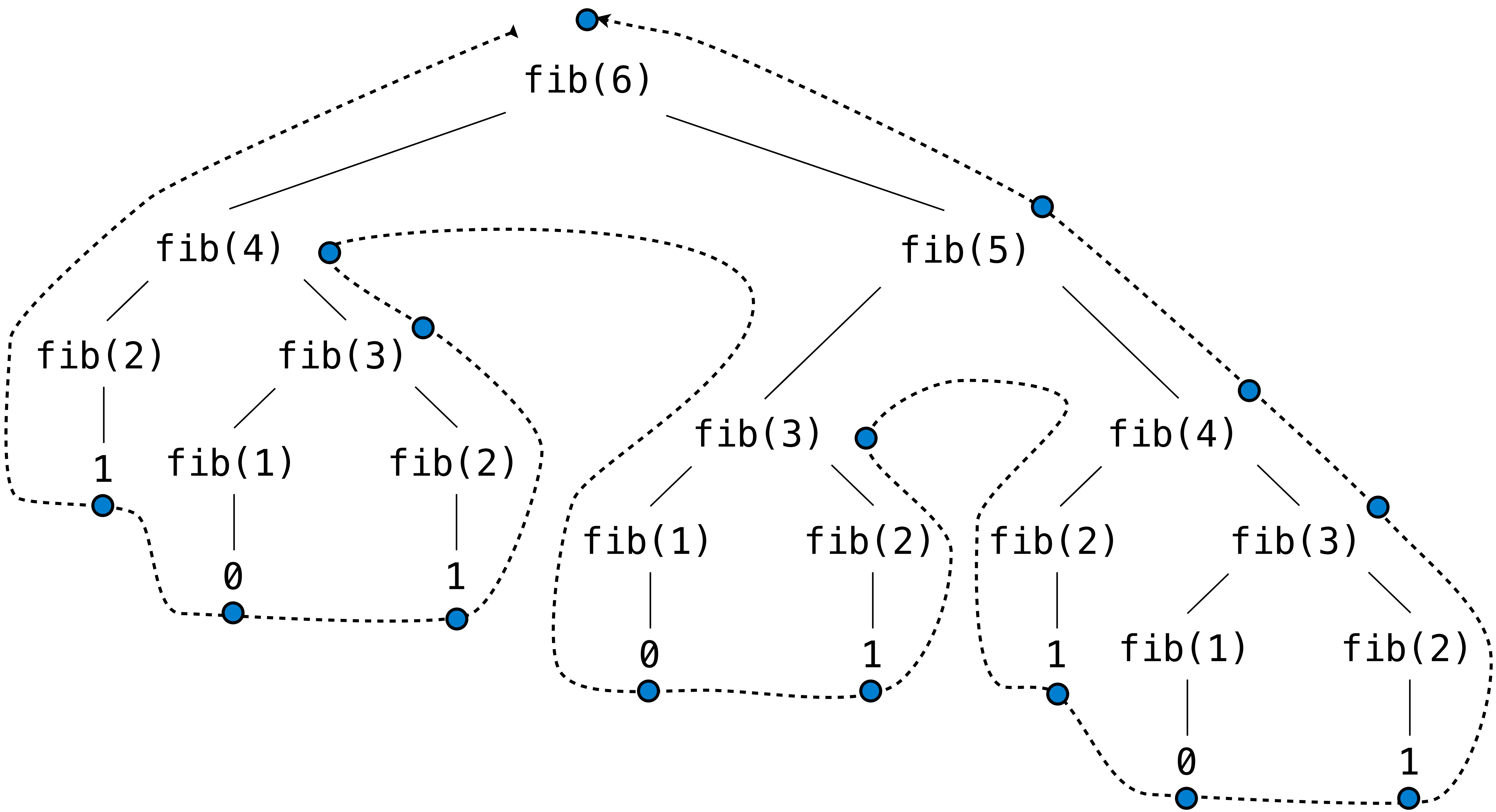

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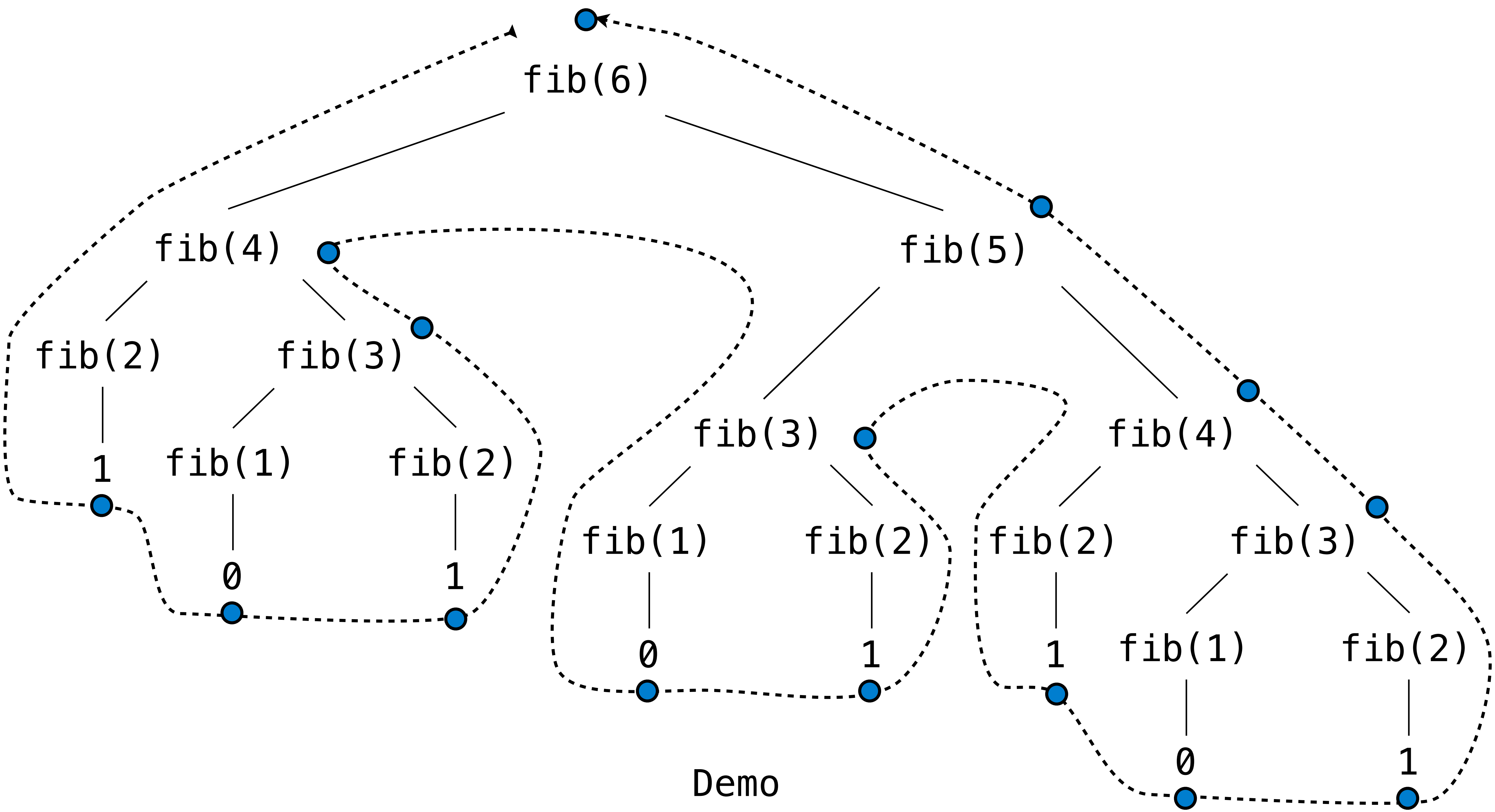

Demo

Repetition in Tree-Recursive Computation

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This process is highly repetitive; fib is called on the same argument multiple times

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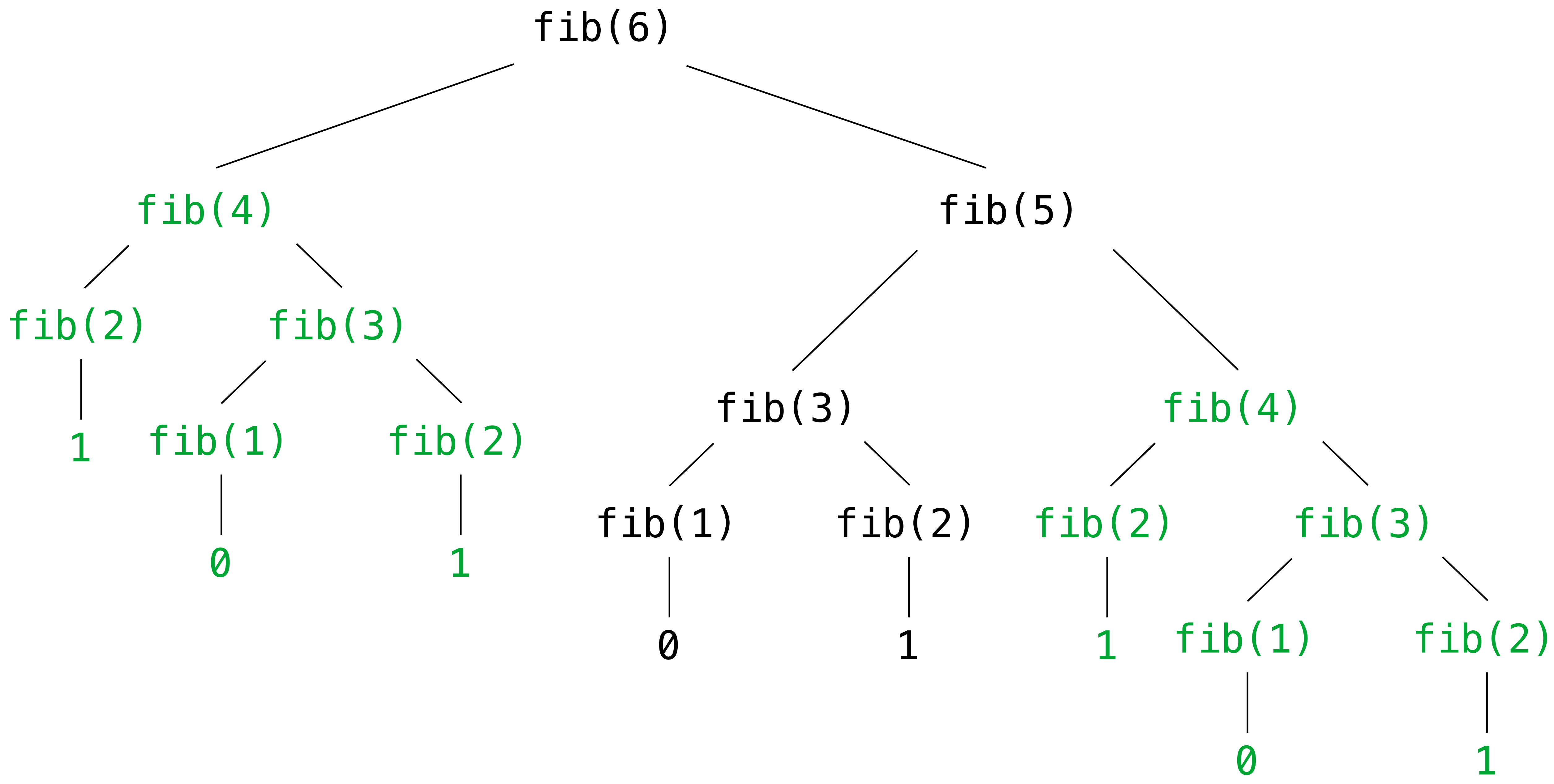

Friday, October 14, 2011

Memoization

Idea: Remember the results that have been computed before

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        if n not in cache:
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def memo(f):
    cache = {}
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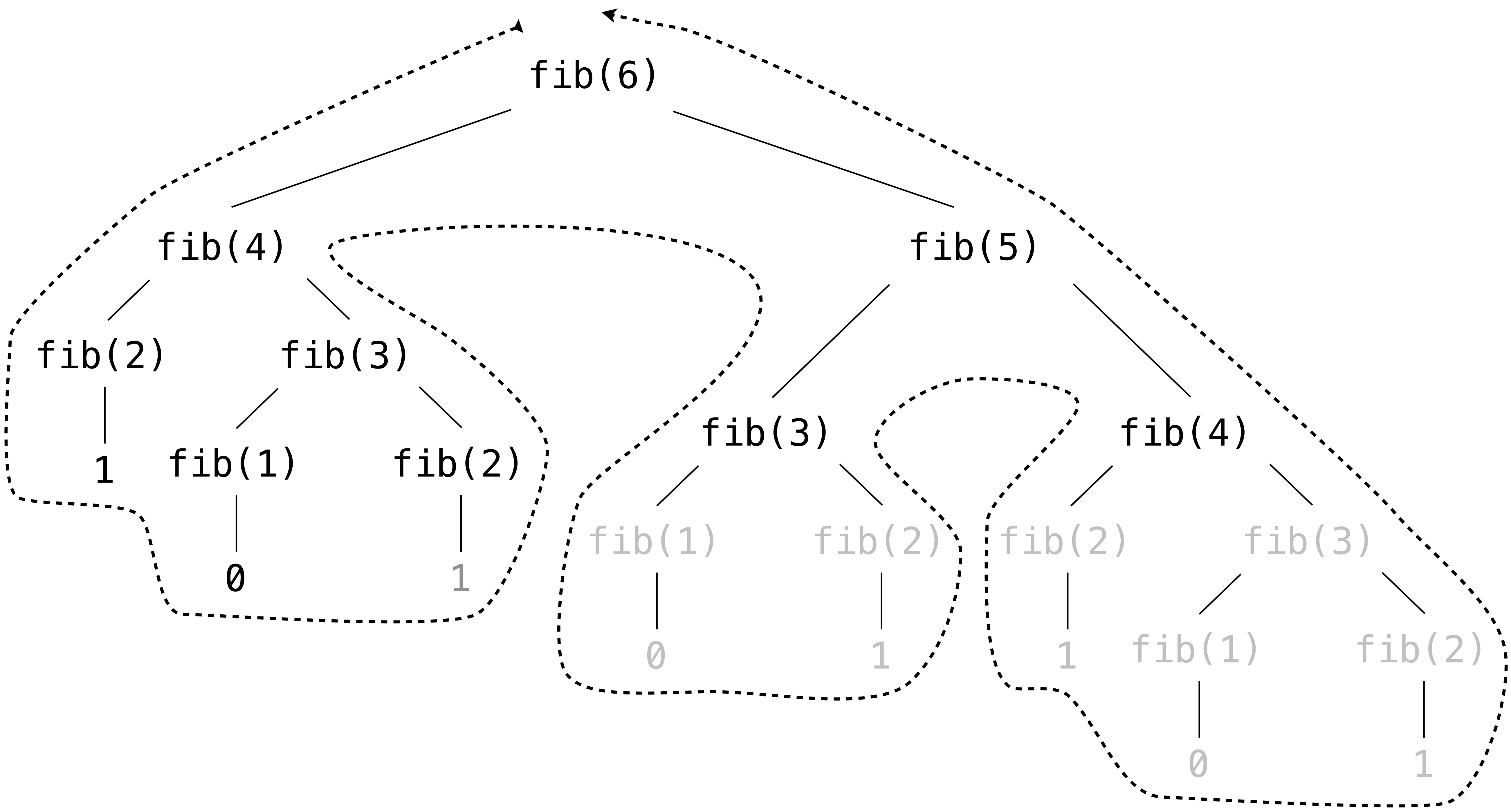
Memoized Tree Recursion



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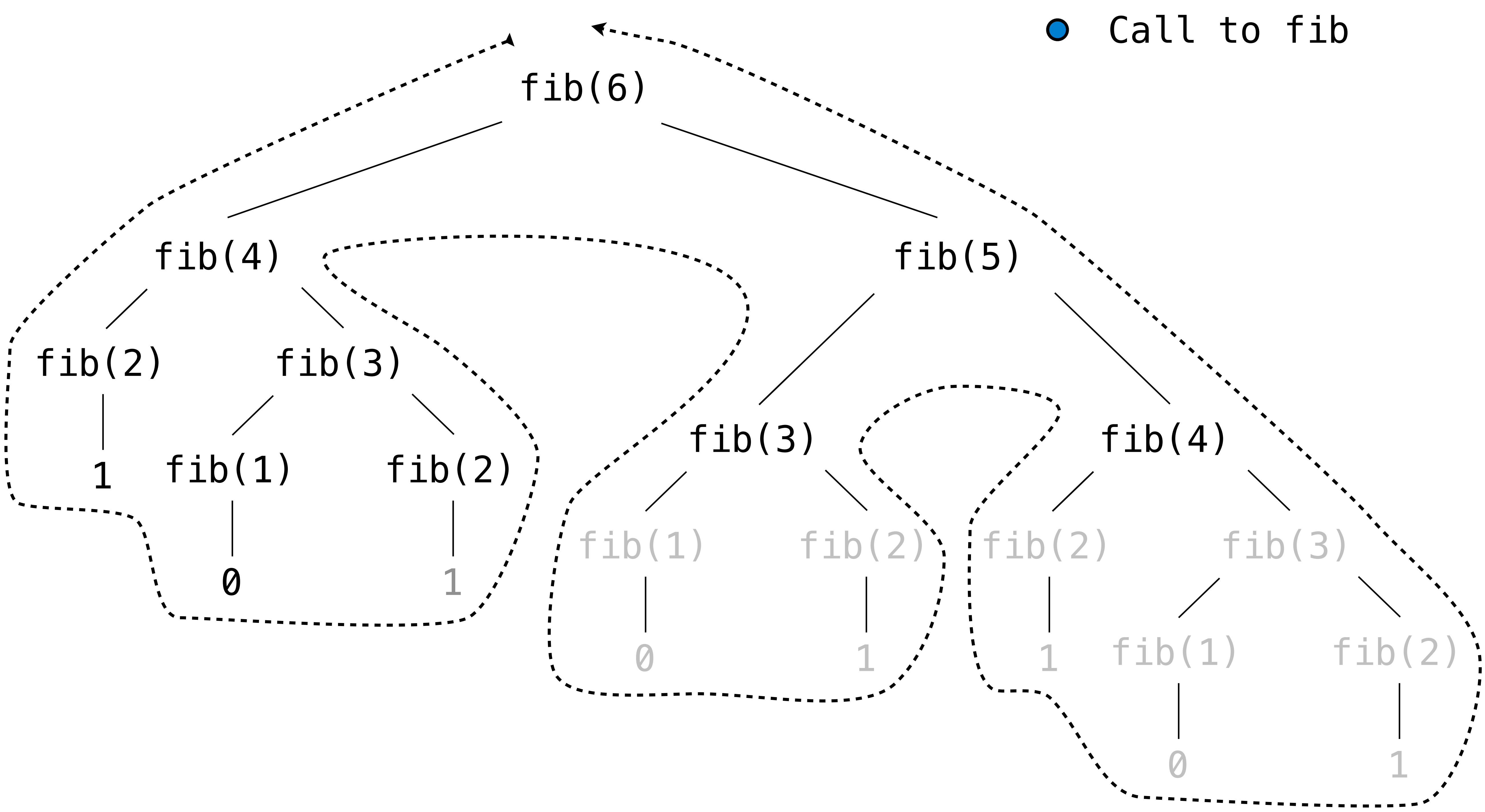

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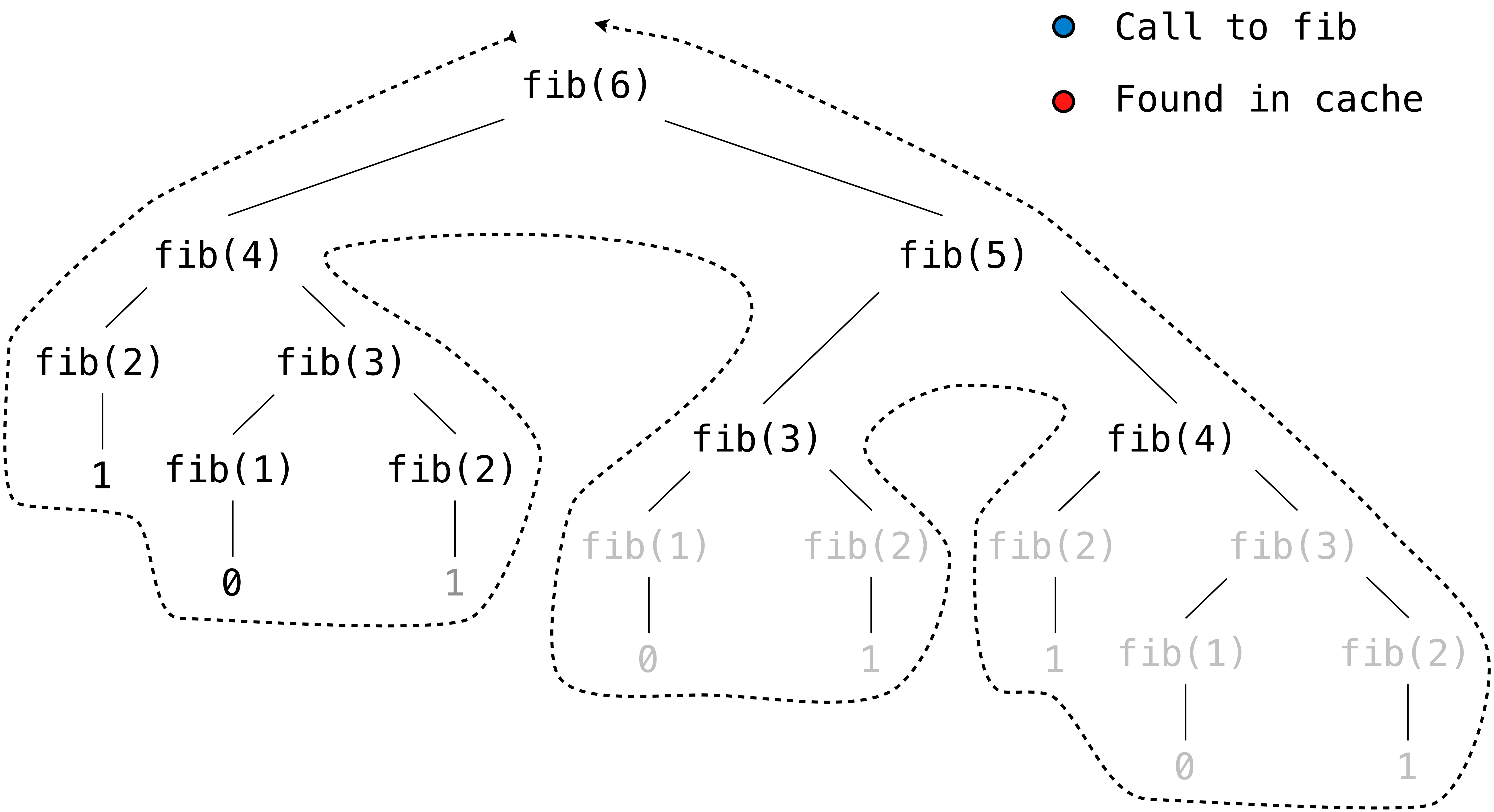

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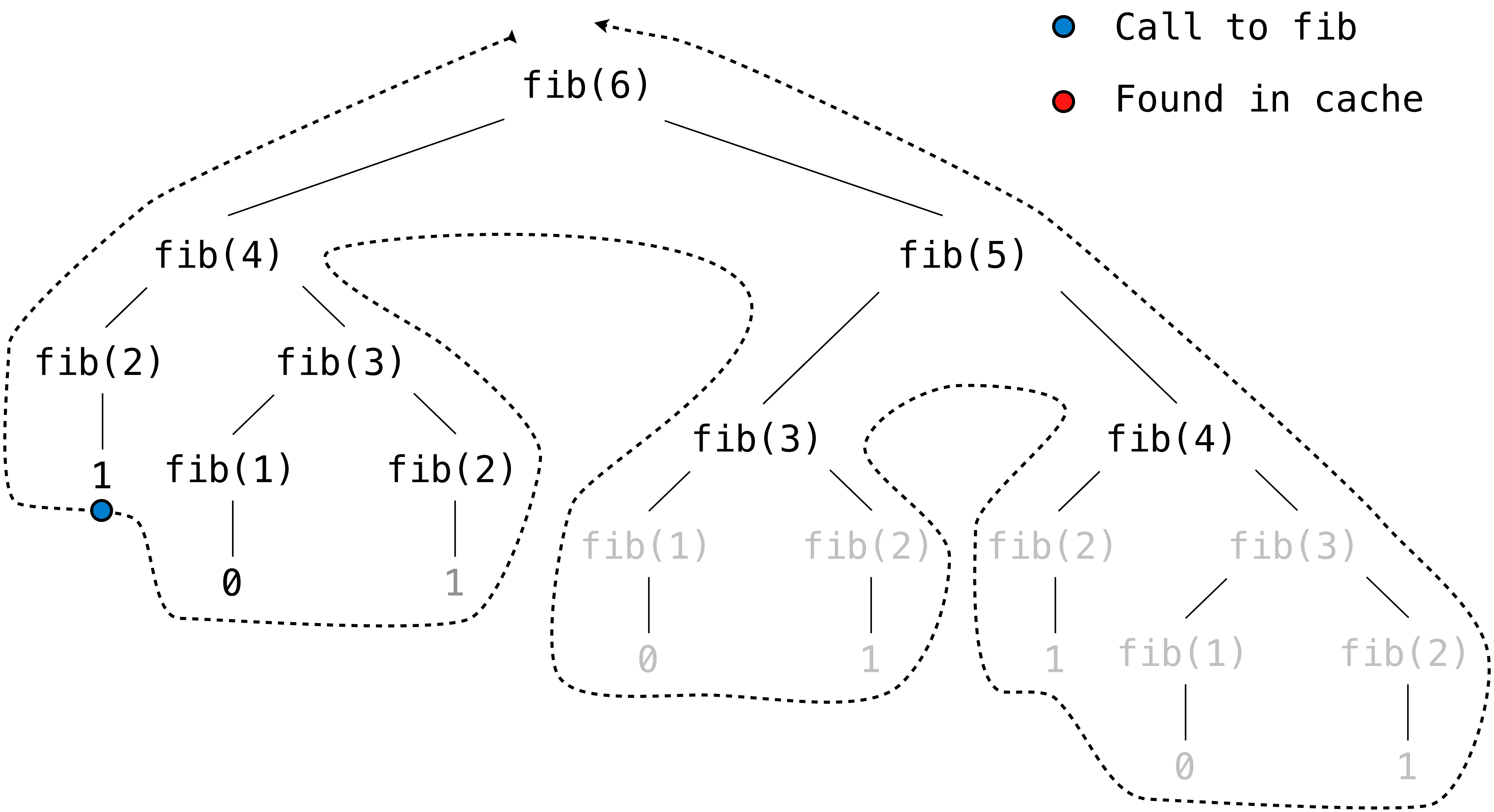

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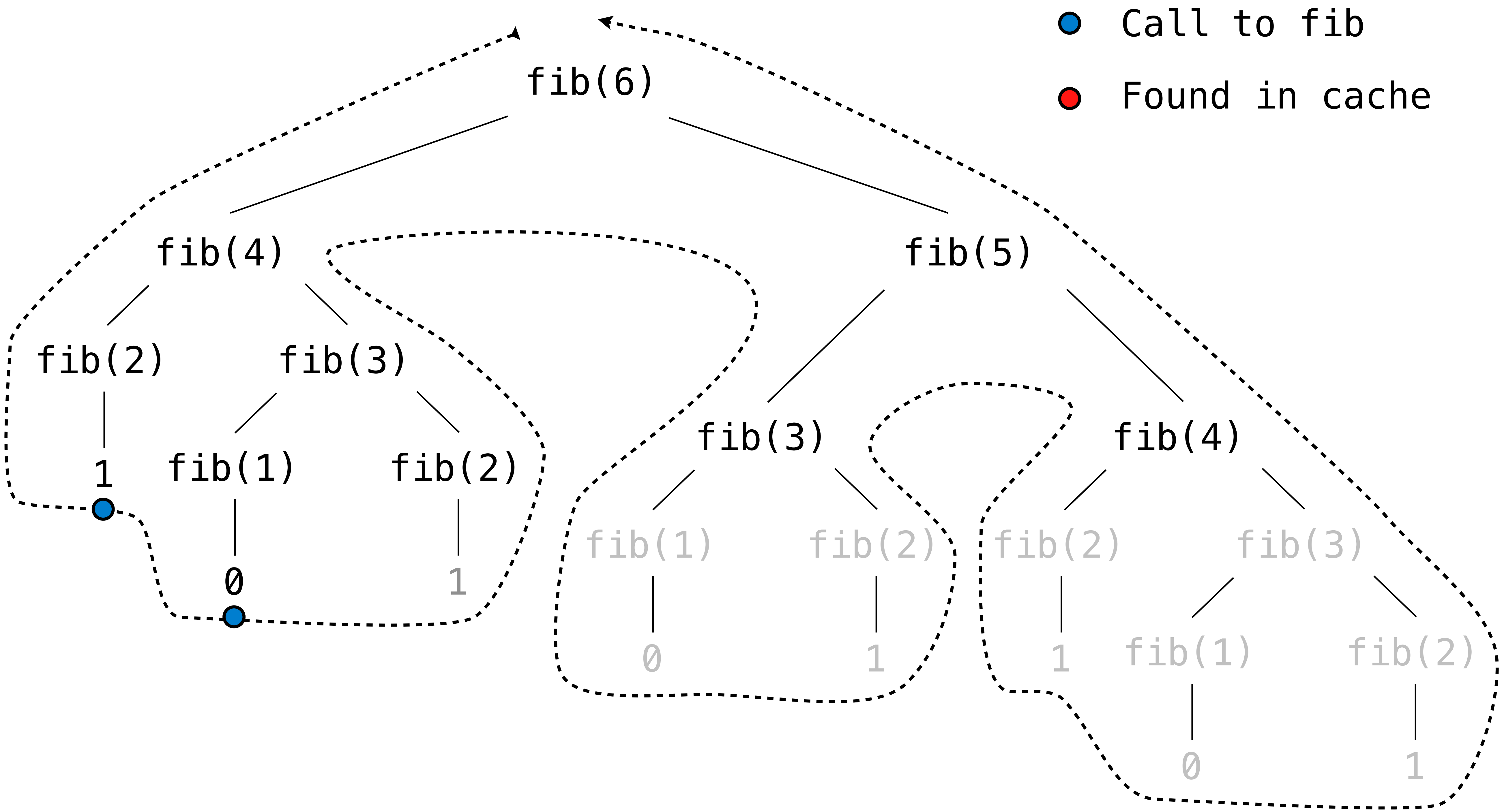

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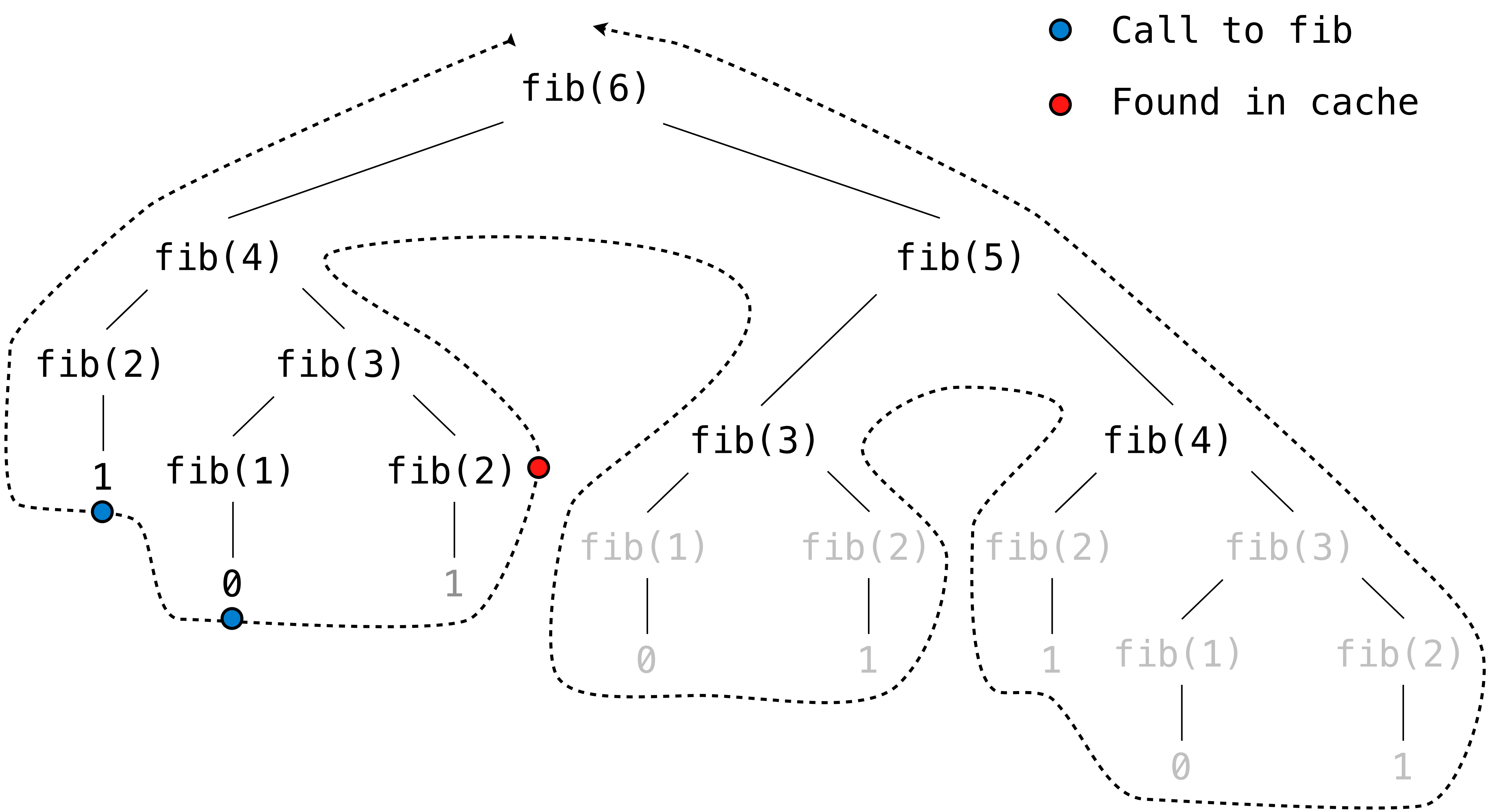

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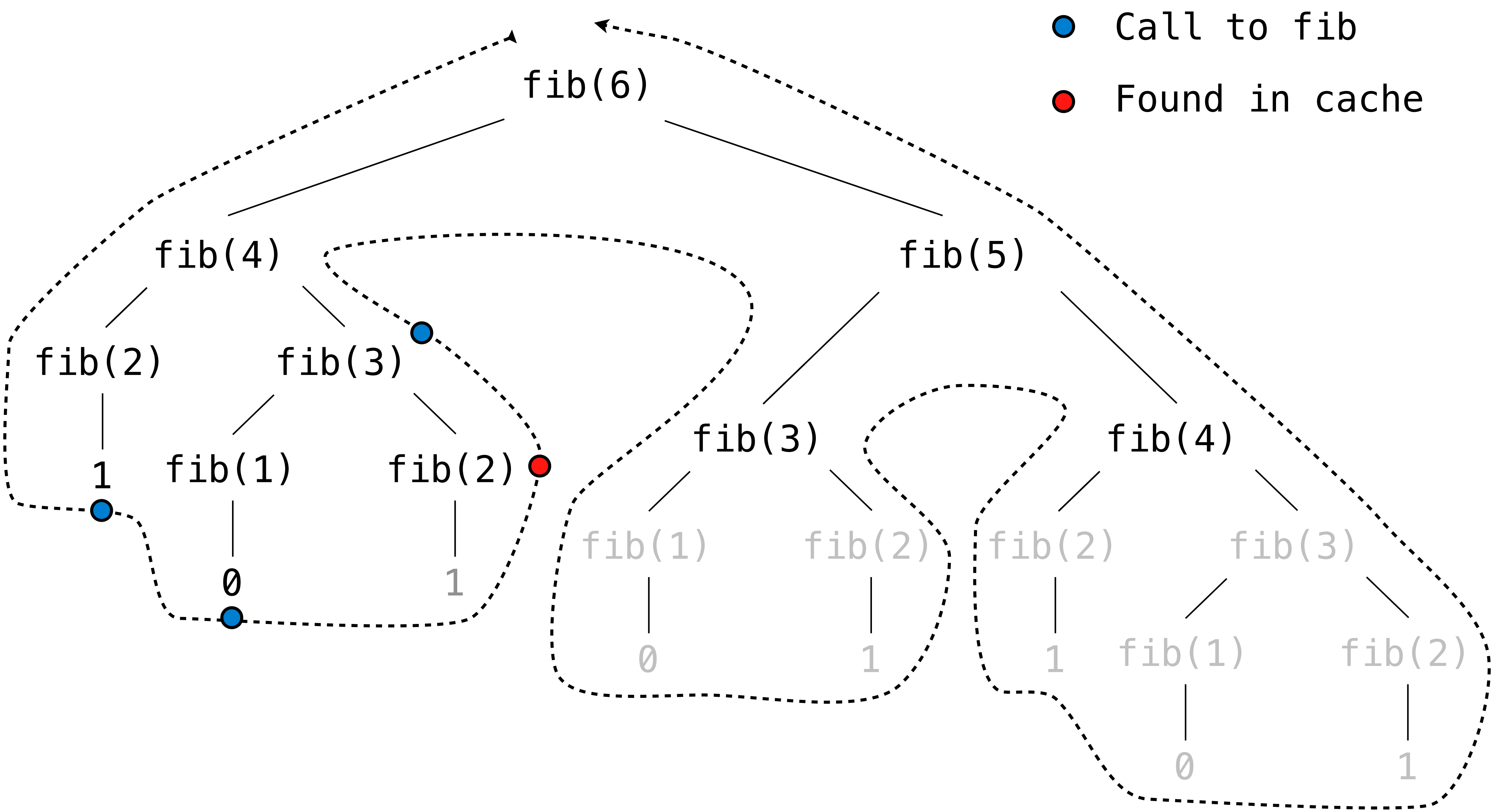

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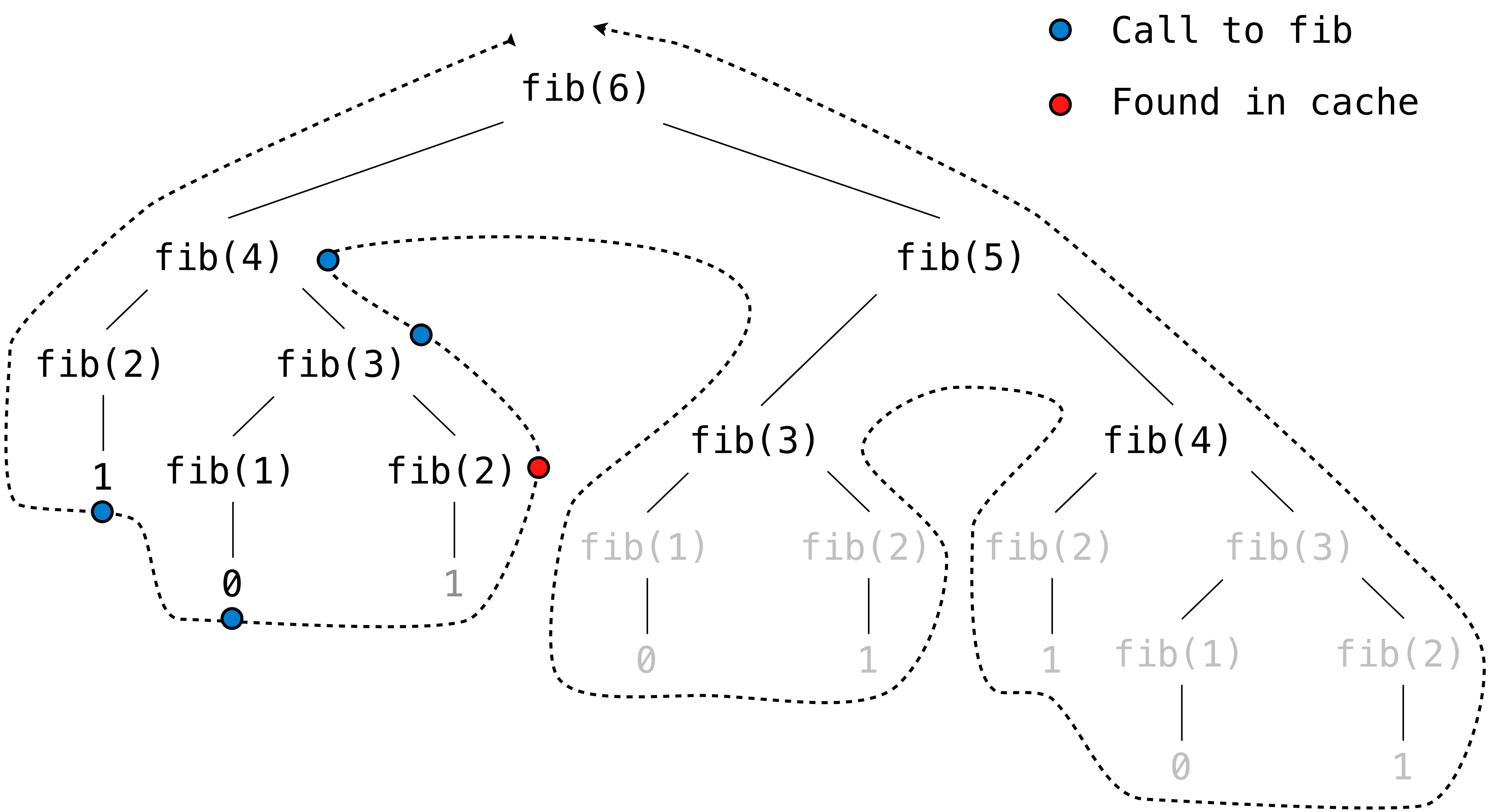

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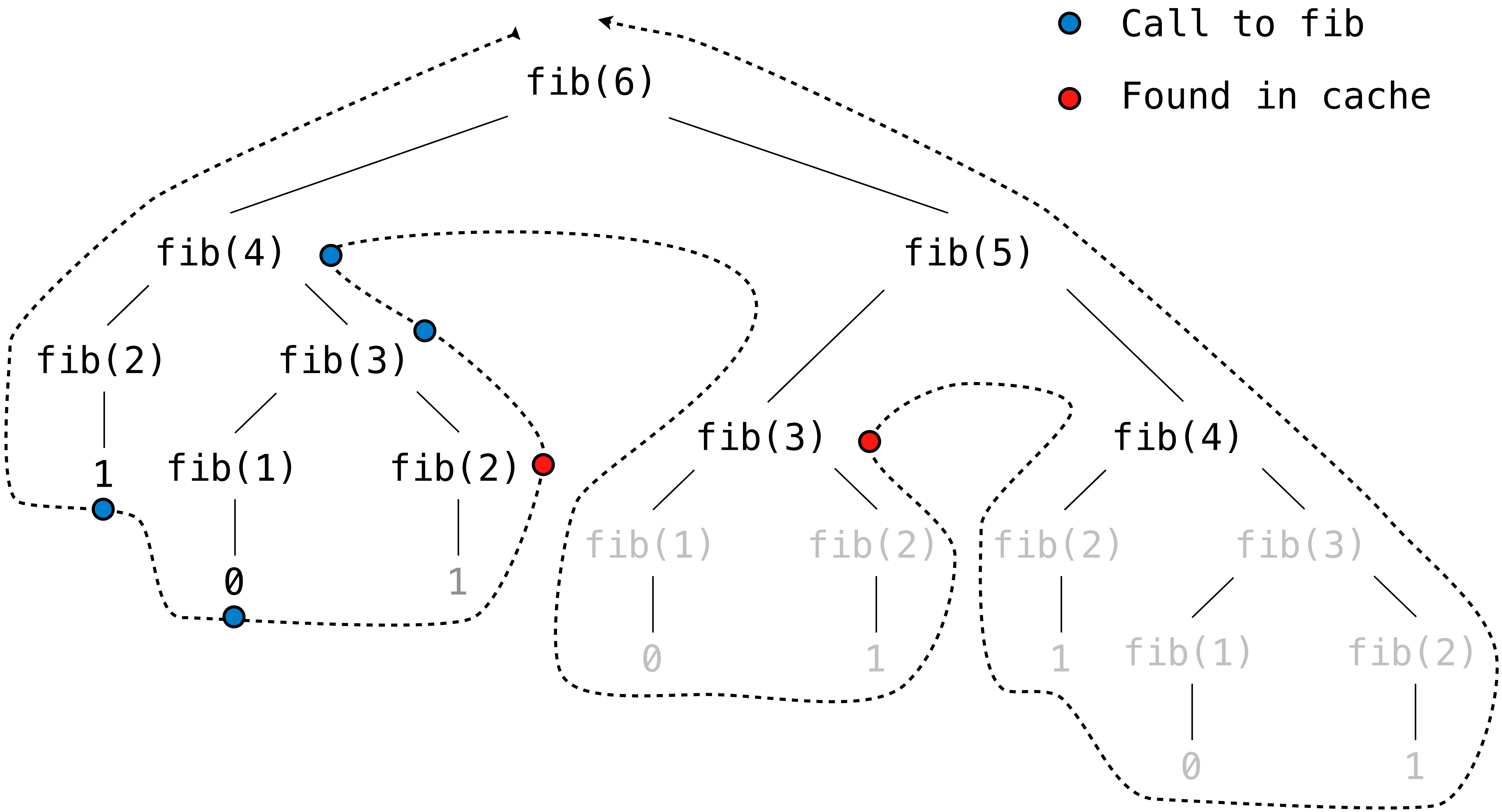

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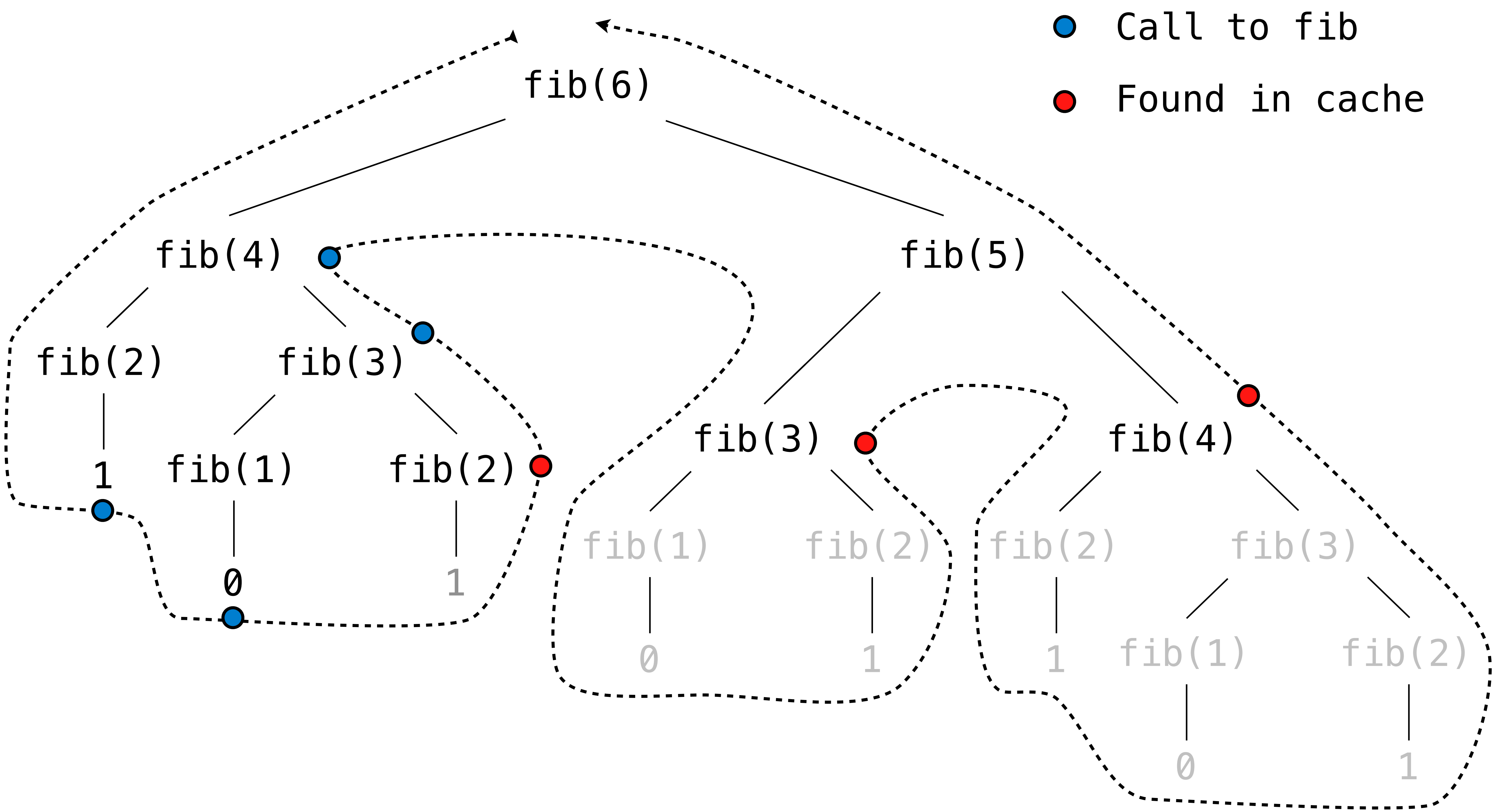

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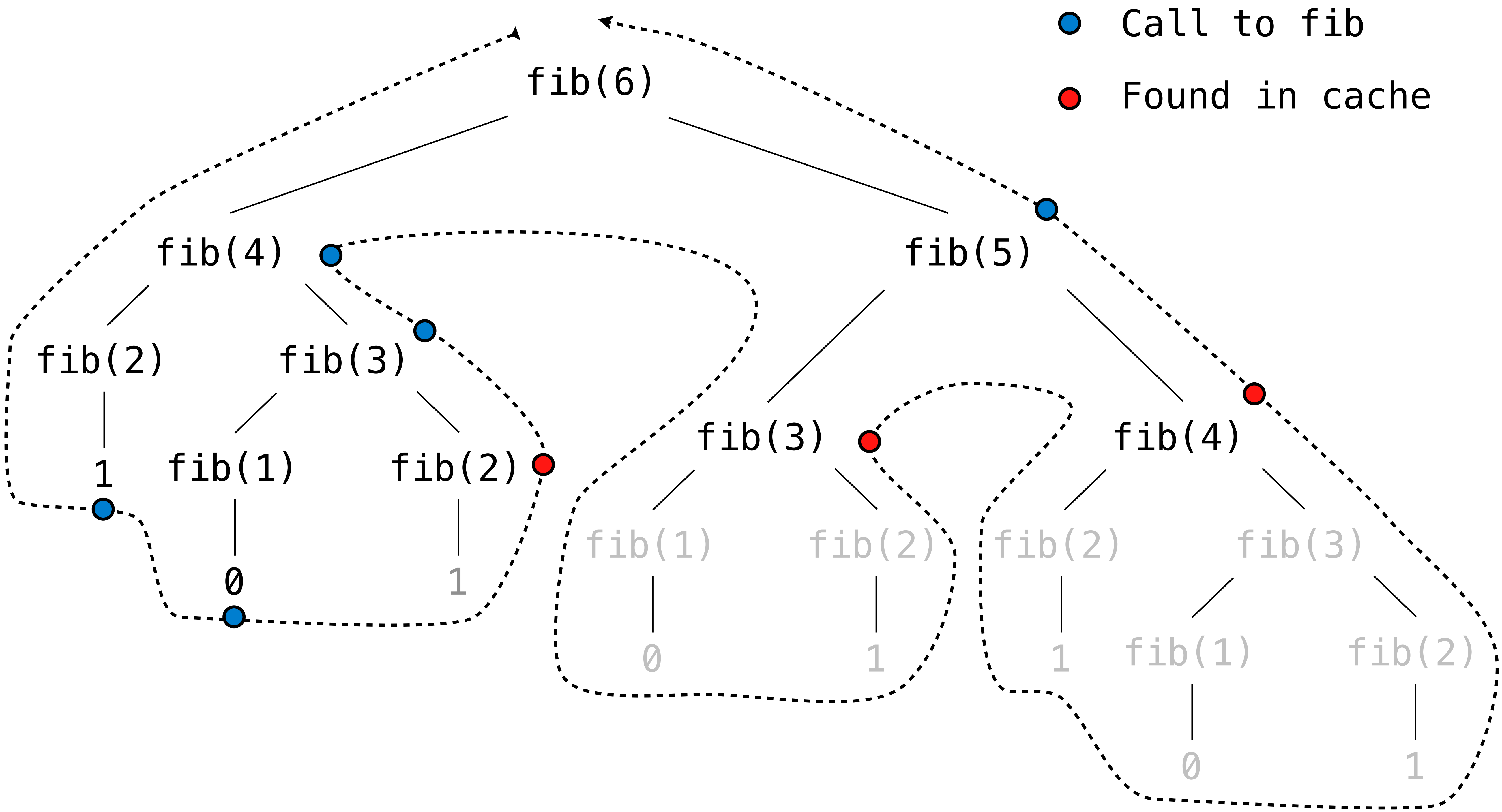

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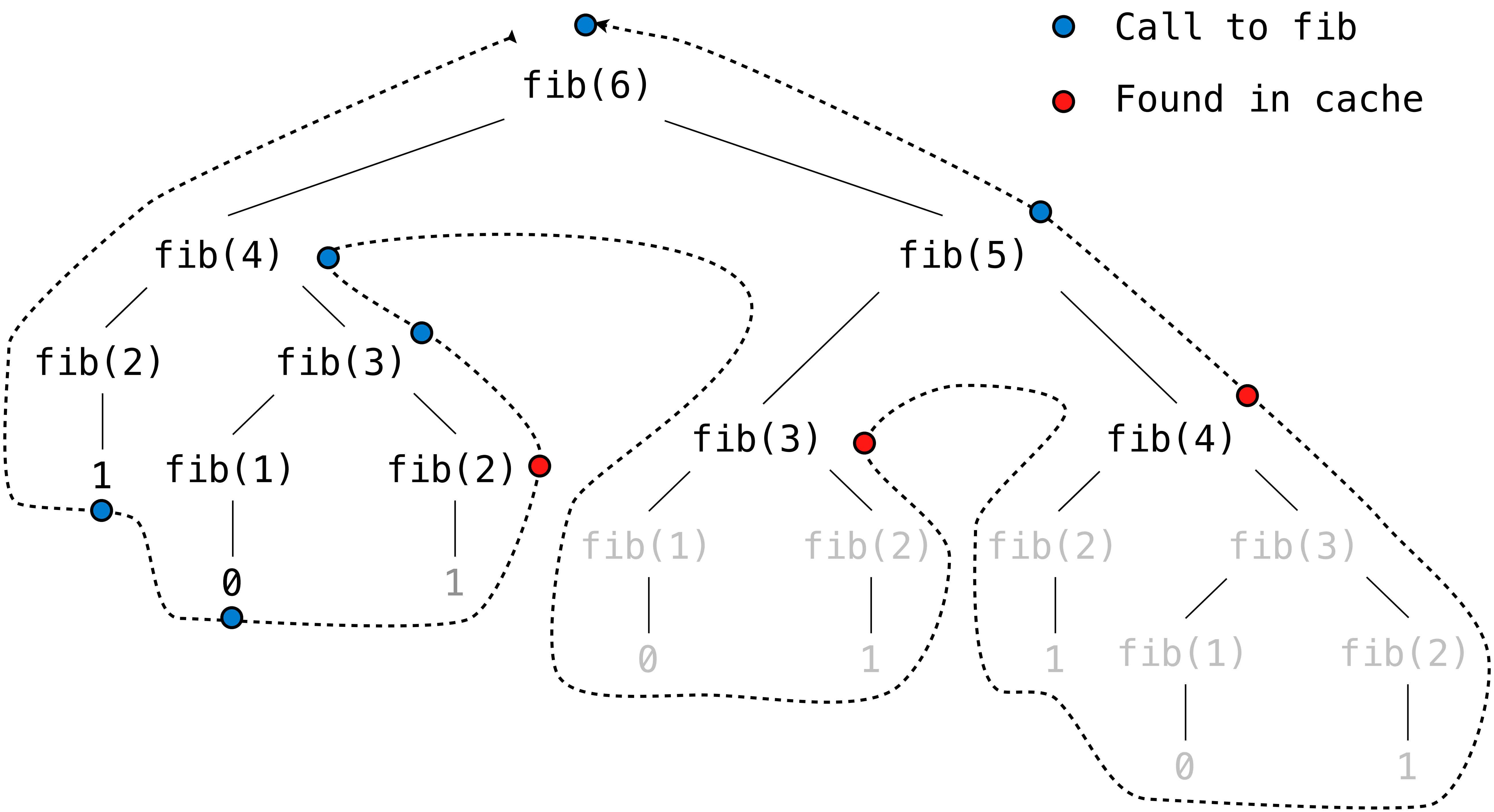

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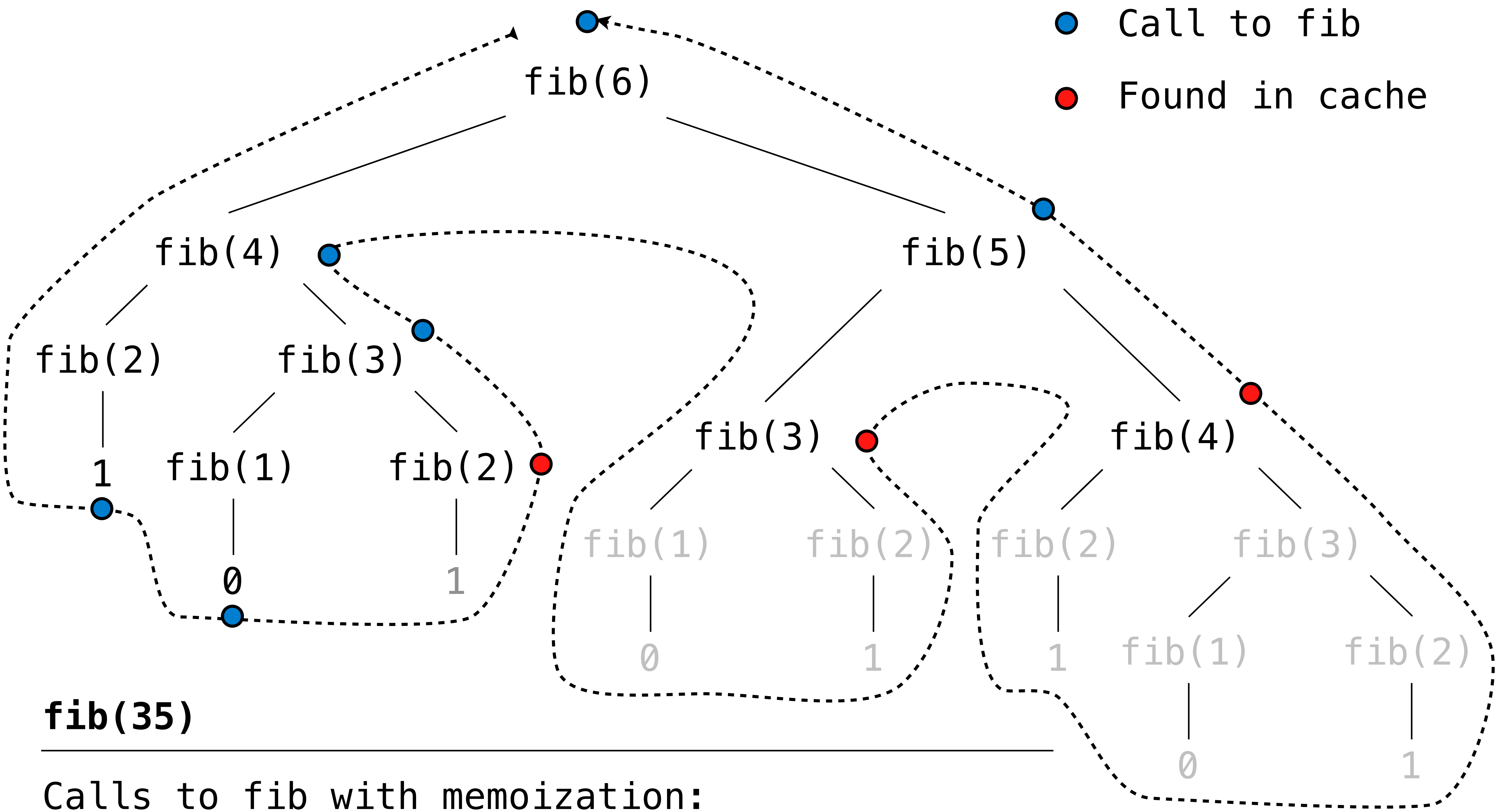
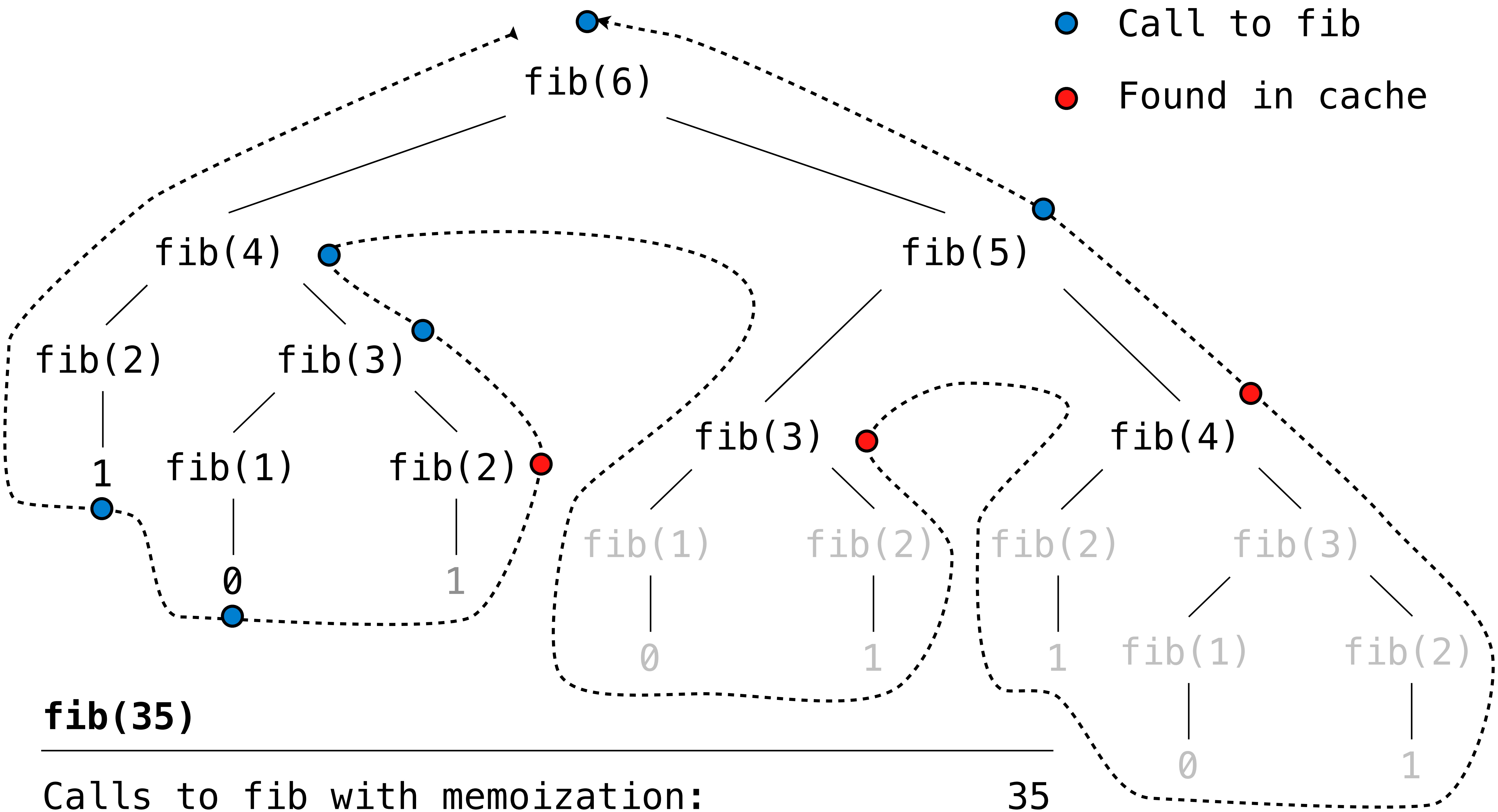


fib(35)

Calls to fib with memoization:

Calls to fib without memoization:

Memoized Tree Recursion



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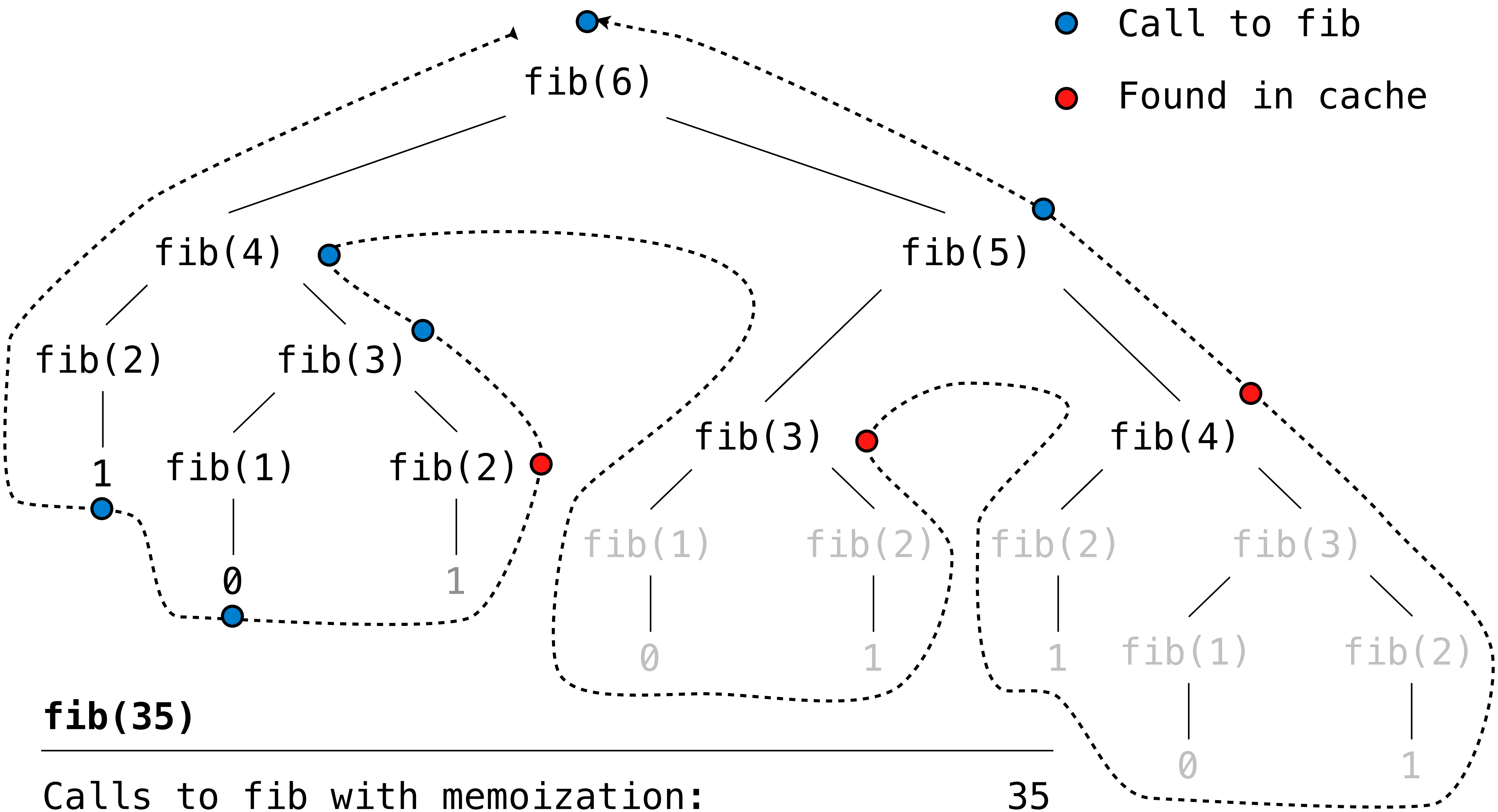

fib(35)

Calls to fib with memoization:	35
Calls to fib without memoization:	18,454,929

Iteration vs Memoized Tree Recursion

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Iterative and memoized implementations are not the same.

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Time	Space
n steps	

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	Time	Space
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fib	n steps	

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	Time	Space
fib_iter	n steps	3 names
fib	n steps	

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	Time	Space
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fib	n steps	n entries

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Time	Space
n steps	3 names

Independent of problem size

```
@memo
def fib(n):
    if n == 1:
        return 0
    if n == 2:
        return 1
    return fib(n-2) + fib(n-1)
```

Time	Space
n steps	n entries

Iteration vs Memoized Tree Recursion

Iterative and memoized implementations are not the same.

The first Fibonacci number

```
def fib_iter(n):
    prev, curr = 1, 0
    for _ in range(n-1):
        prev, curr = curr, prev + curr
    return curr
```

Time	Space
n steps	3 names

Independent of problem size

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Scales with problem size

Counting Change

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$1 = $0.50 + $0.25 + $0.10 + $0.10 + $0.05
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```
How many ways are there to change a dollar?
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How many ways to change $0.11 with nickels & pennies?

$0.11 can be changed with nickels & pennies by

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B. Using at least one nickel; $0.06 with nickels & pennies

Counting Change Recursively

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Demo

Space Consumption

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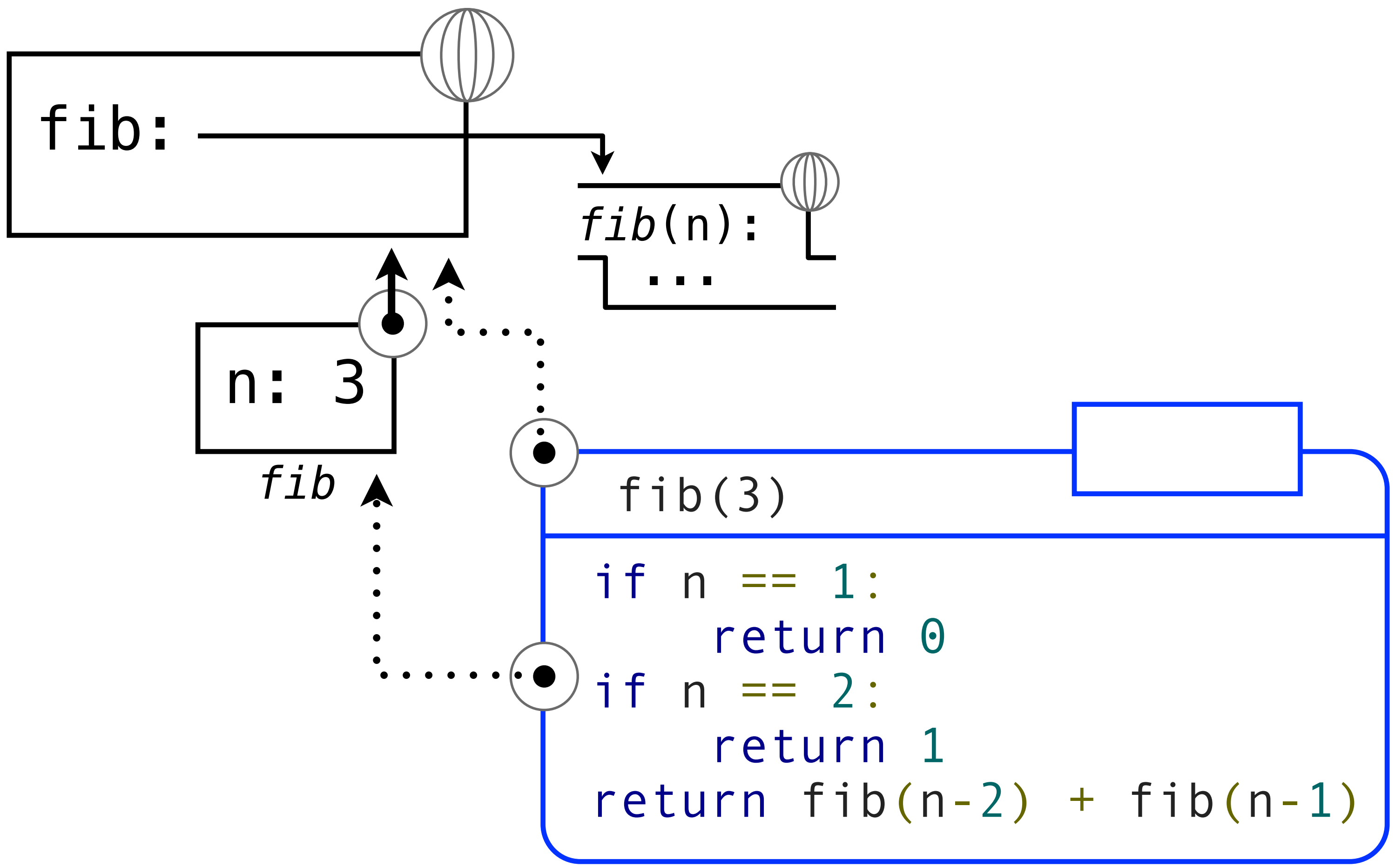
- The environment for the current expression being evaluated
- All environments for expressions that depend upon the value of the current expression
- All environments associated with values referenced by active environments

Fibonacci Environment Diagram



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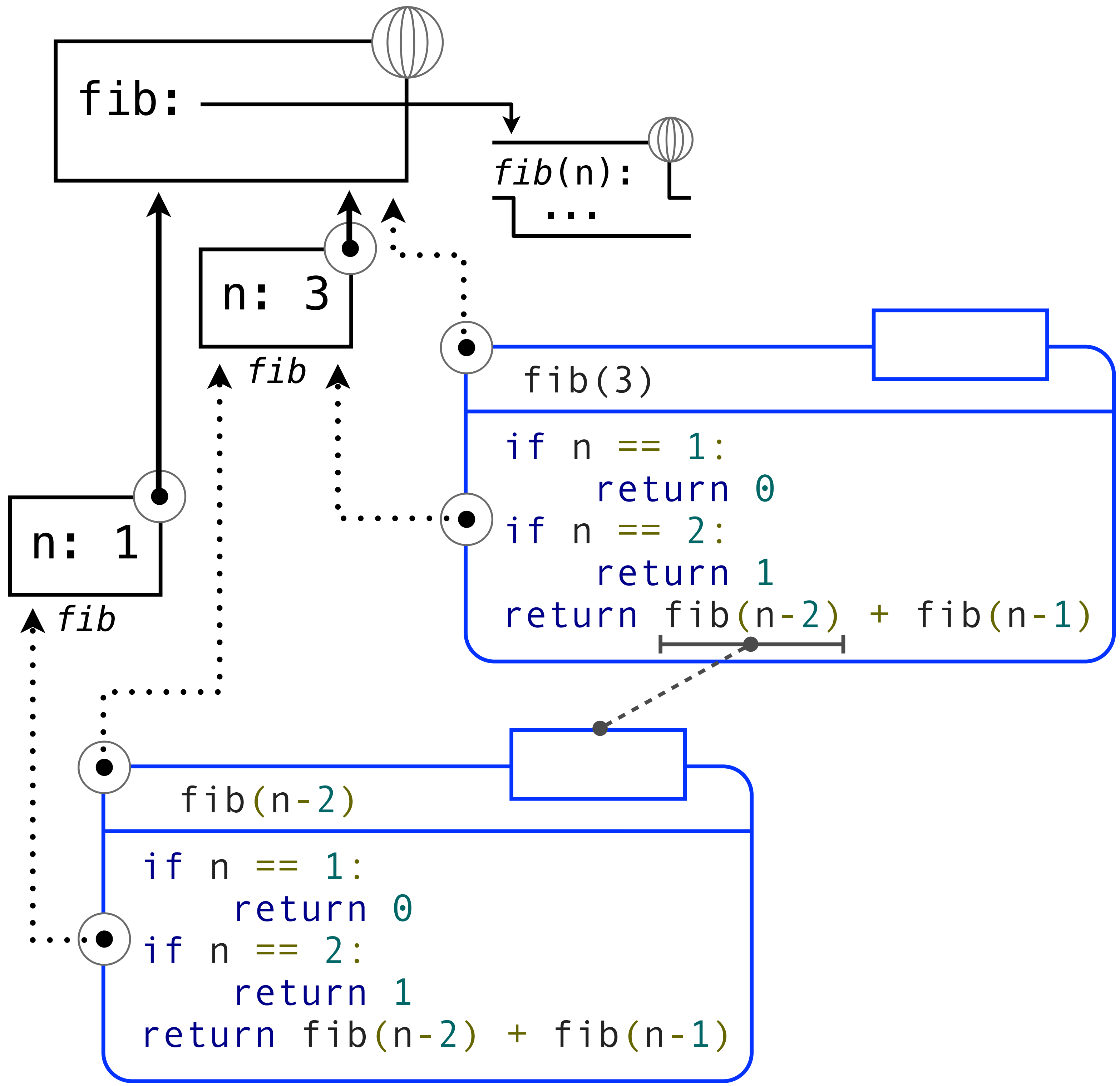

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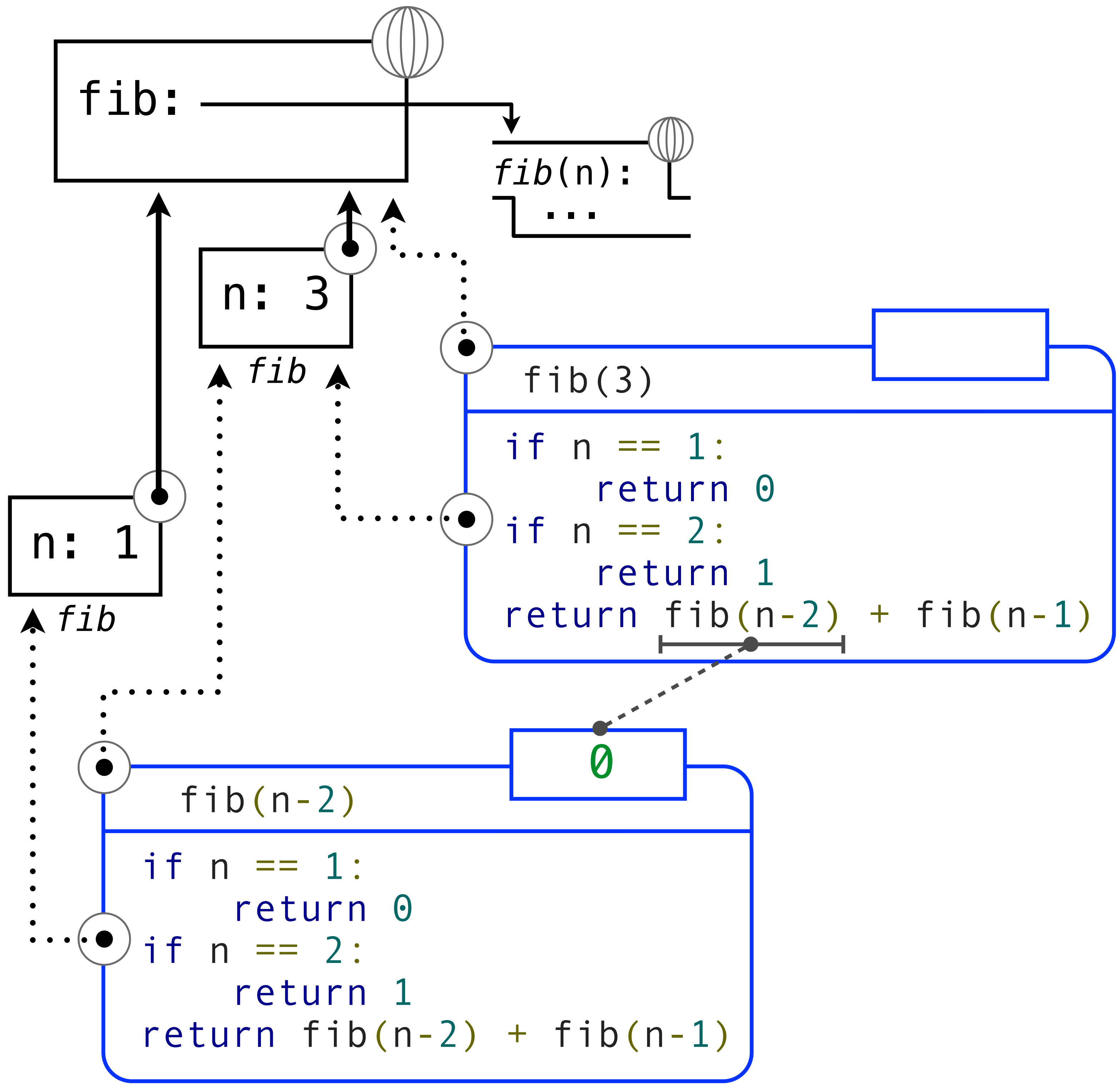

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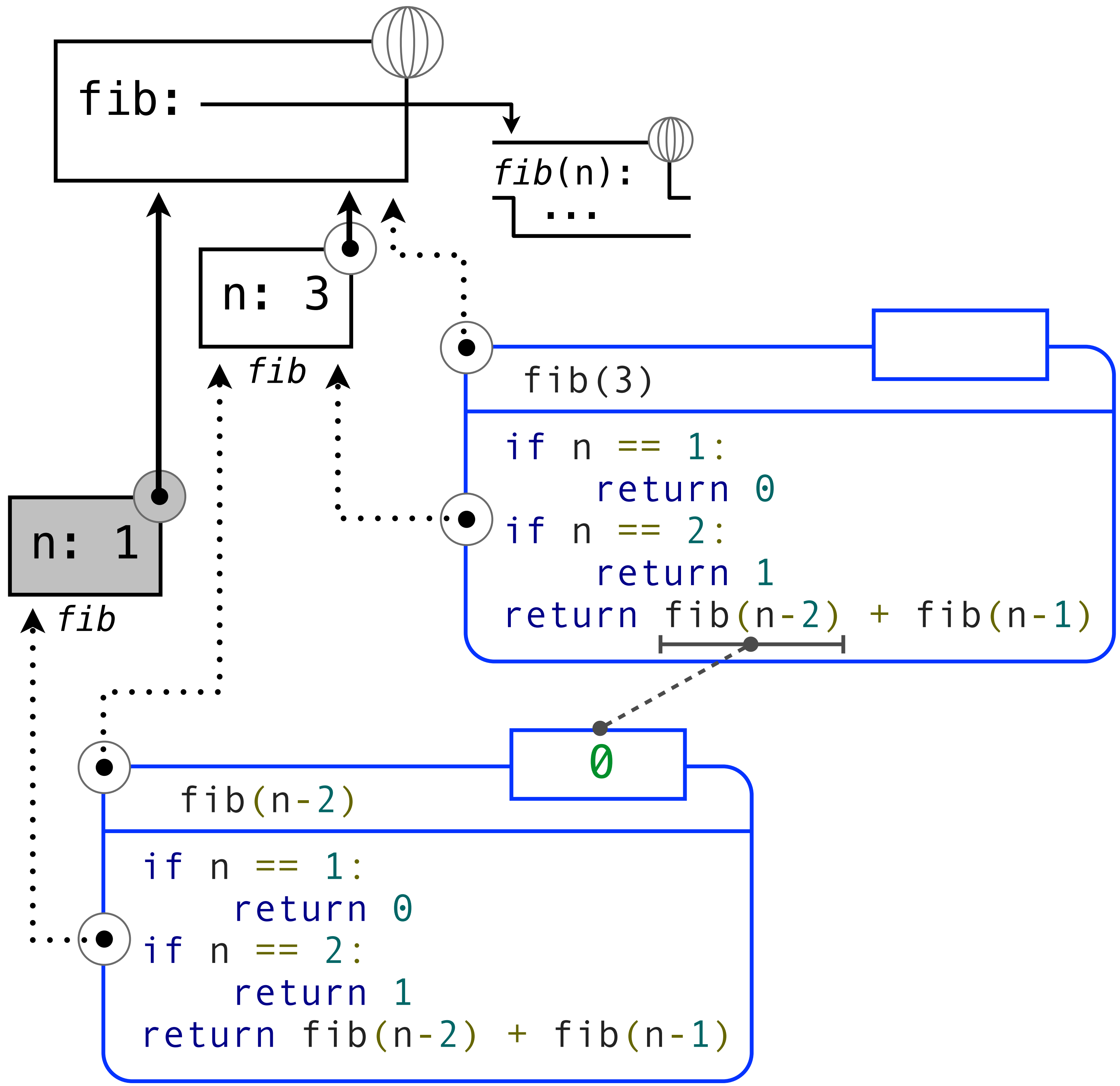

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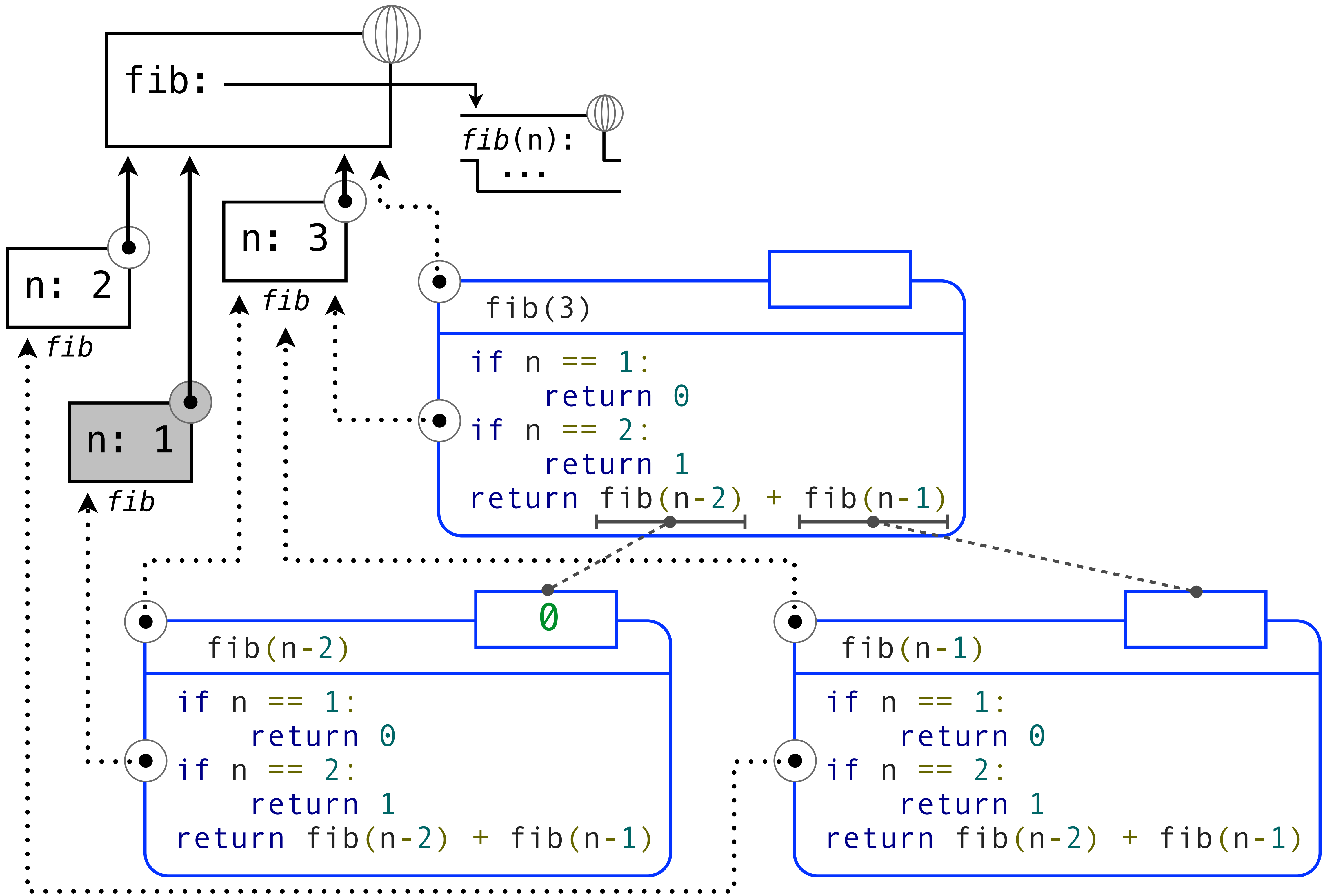
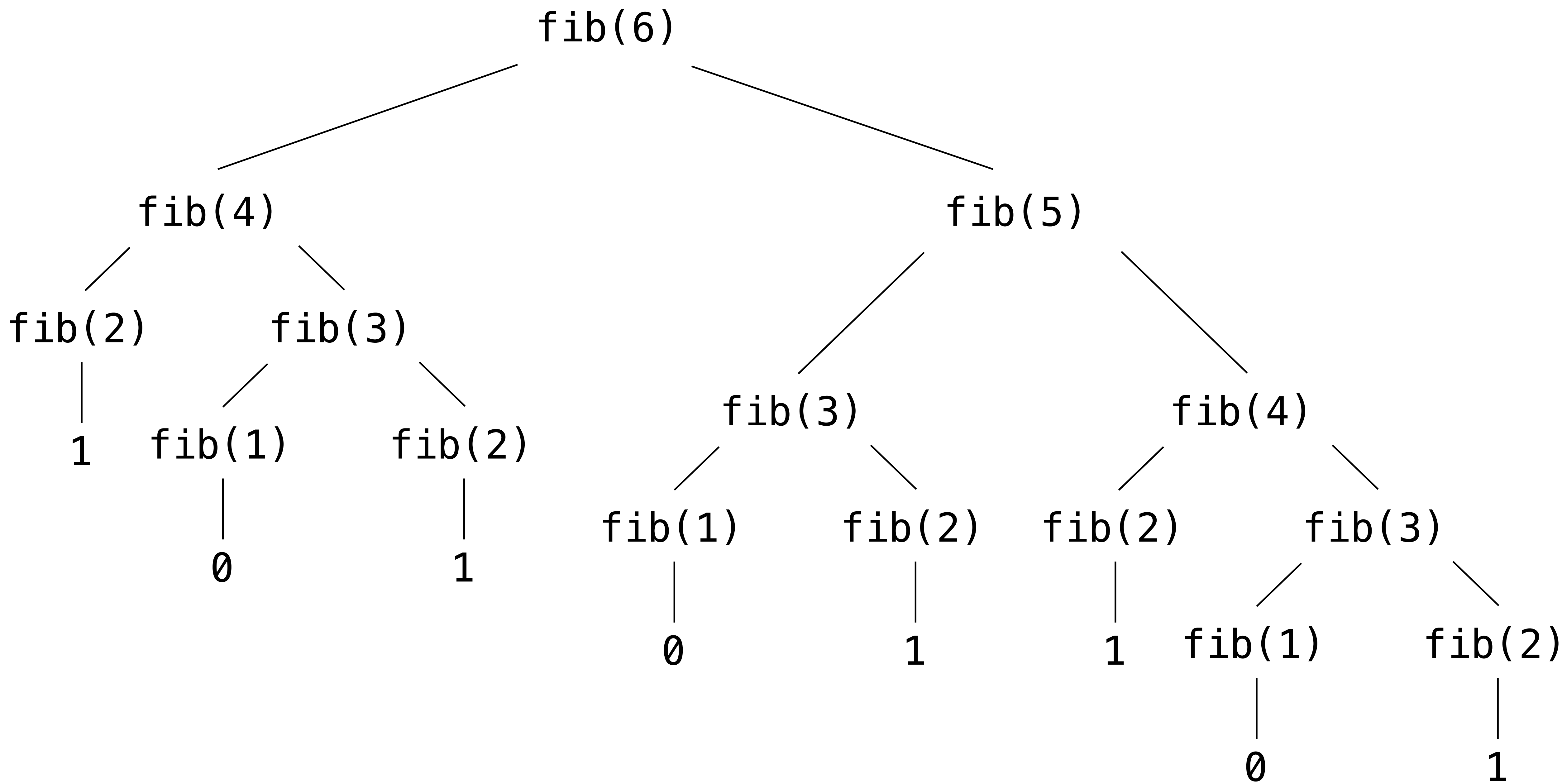


Fibonacci Memory Consumption



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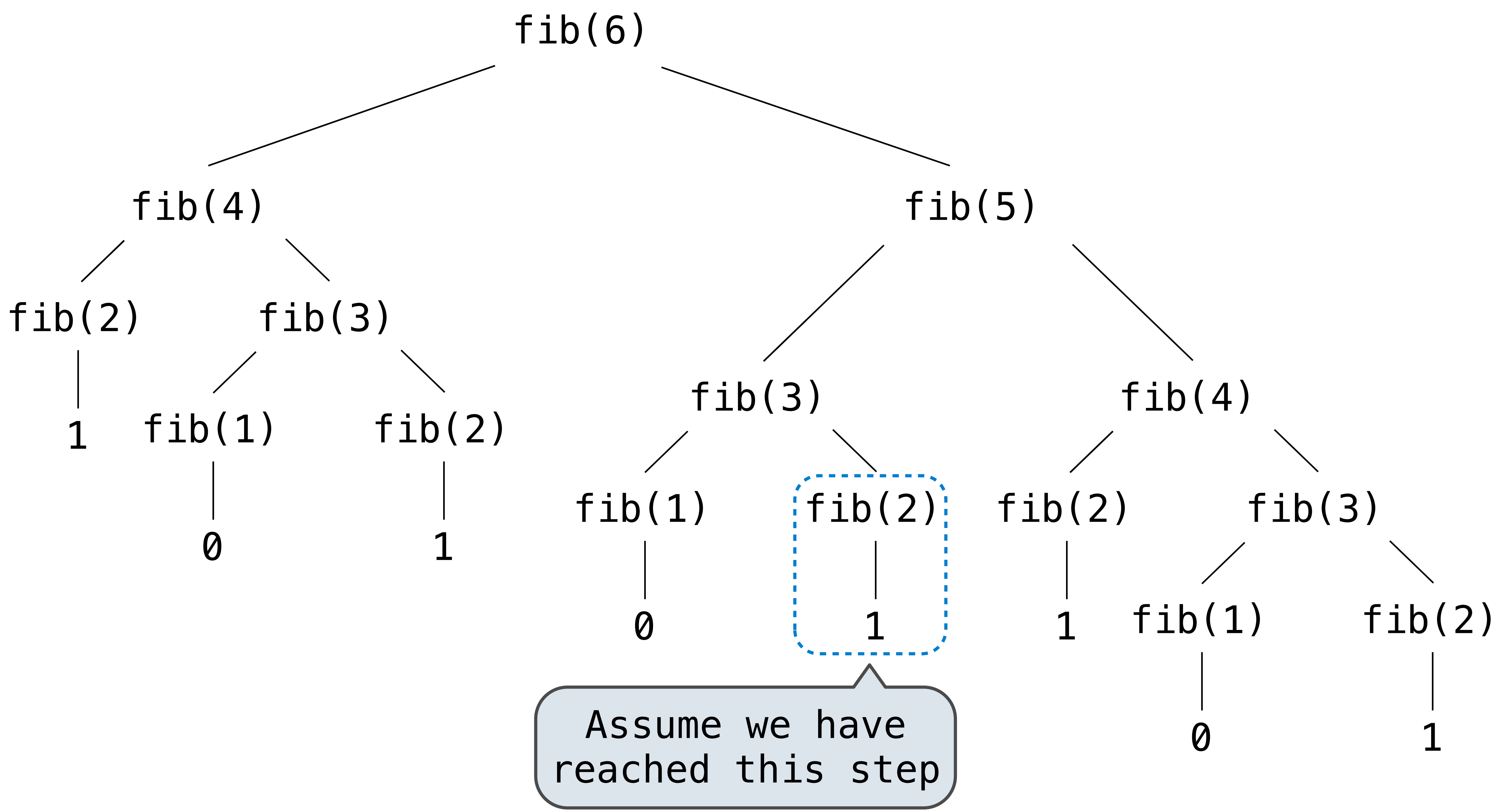

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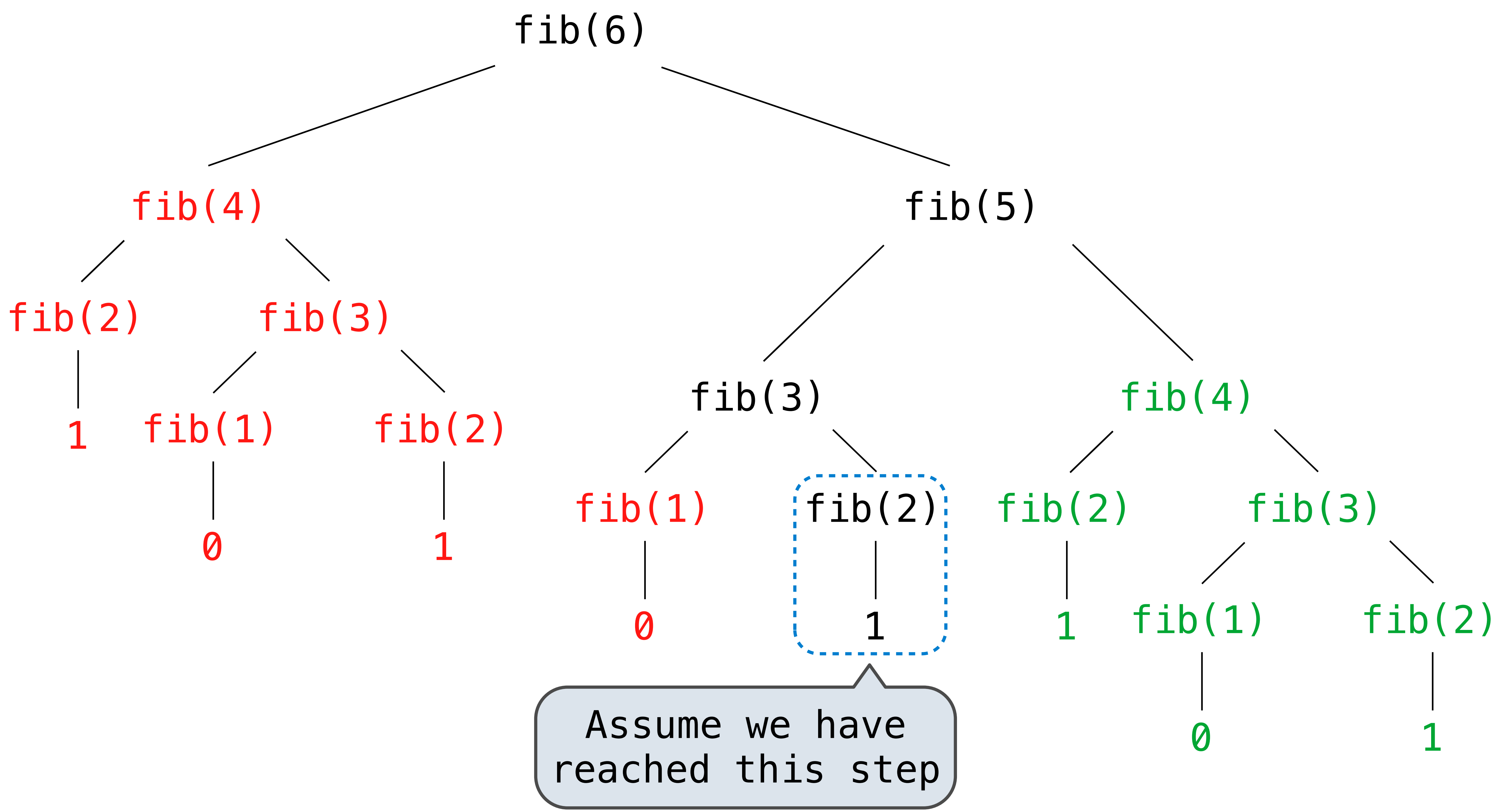

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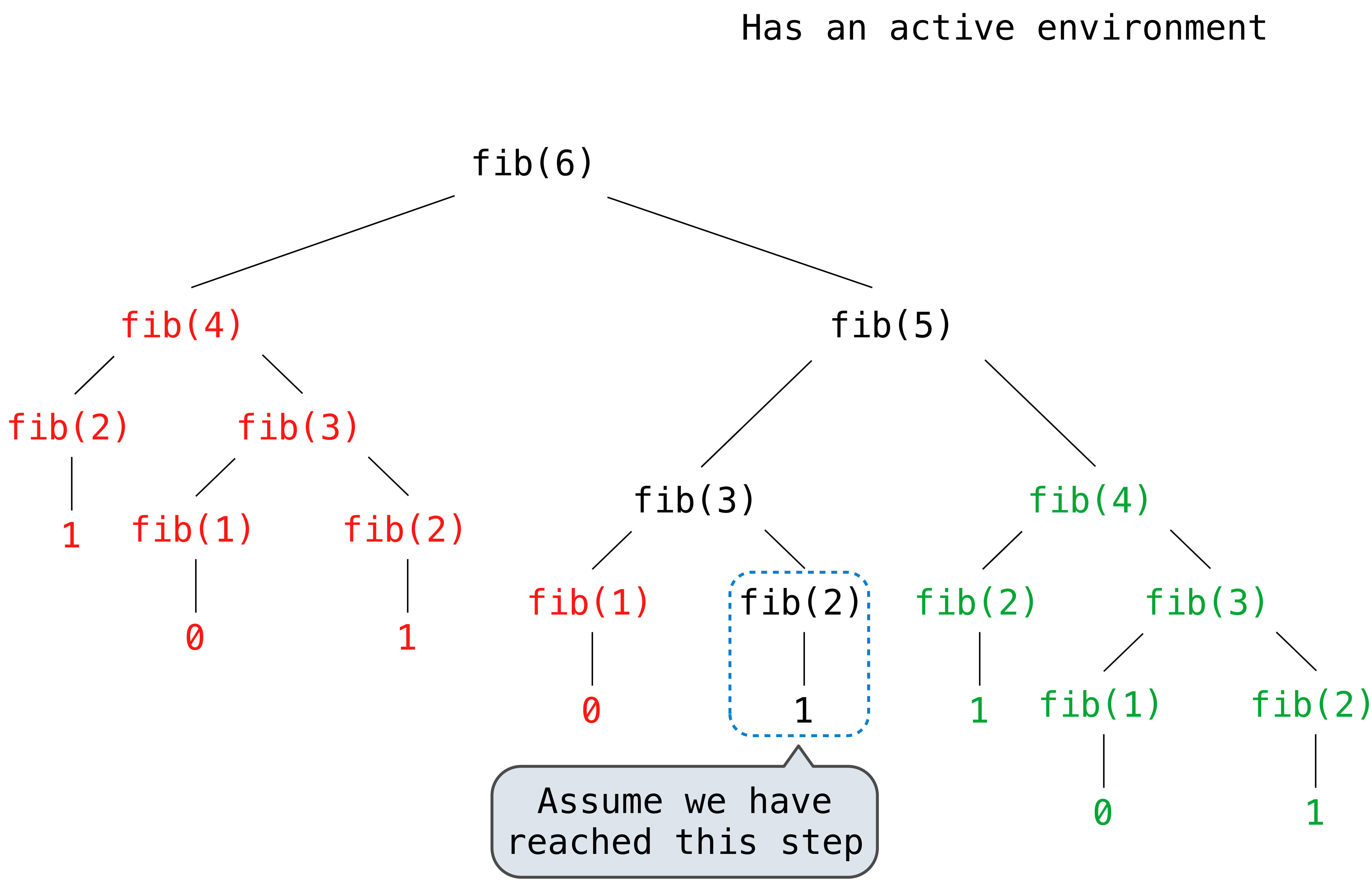

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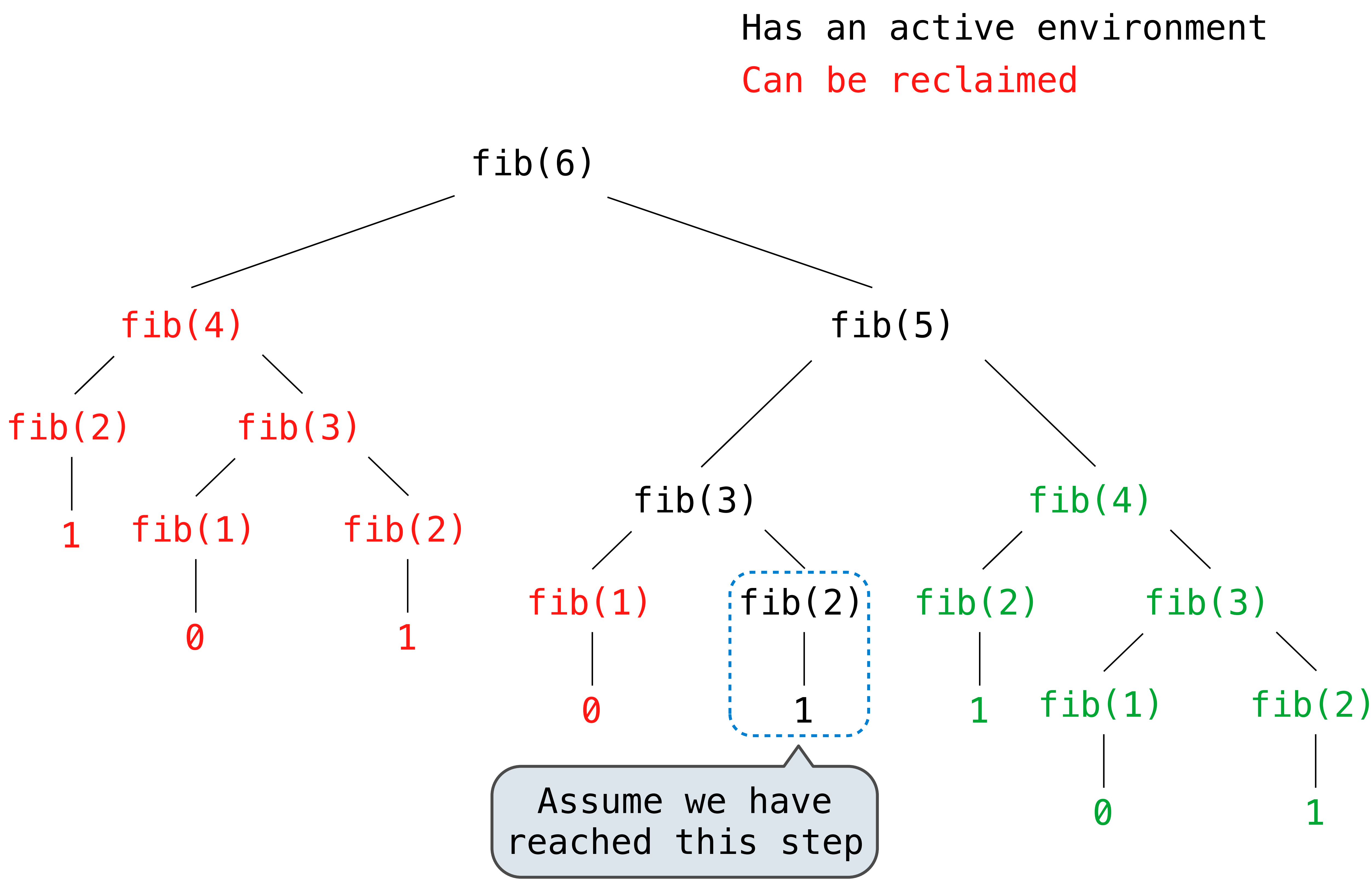

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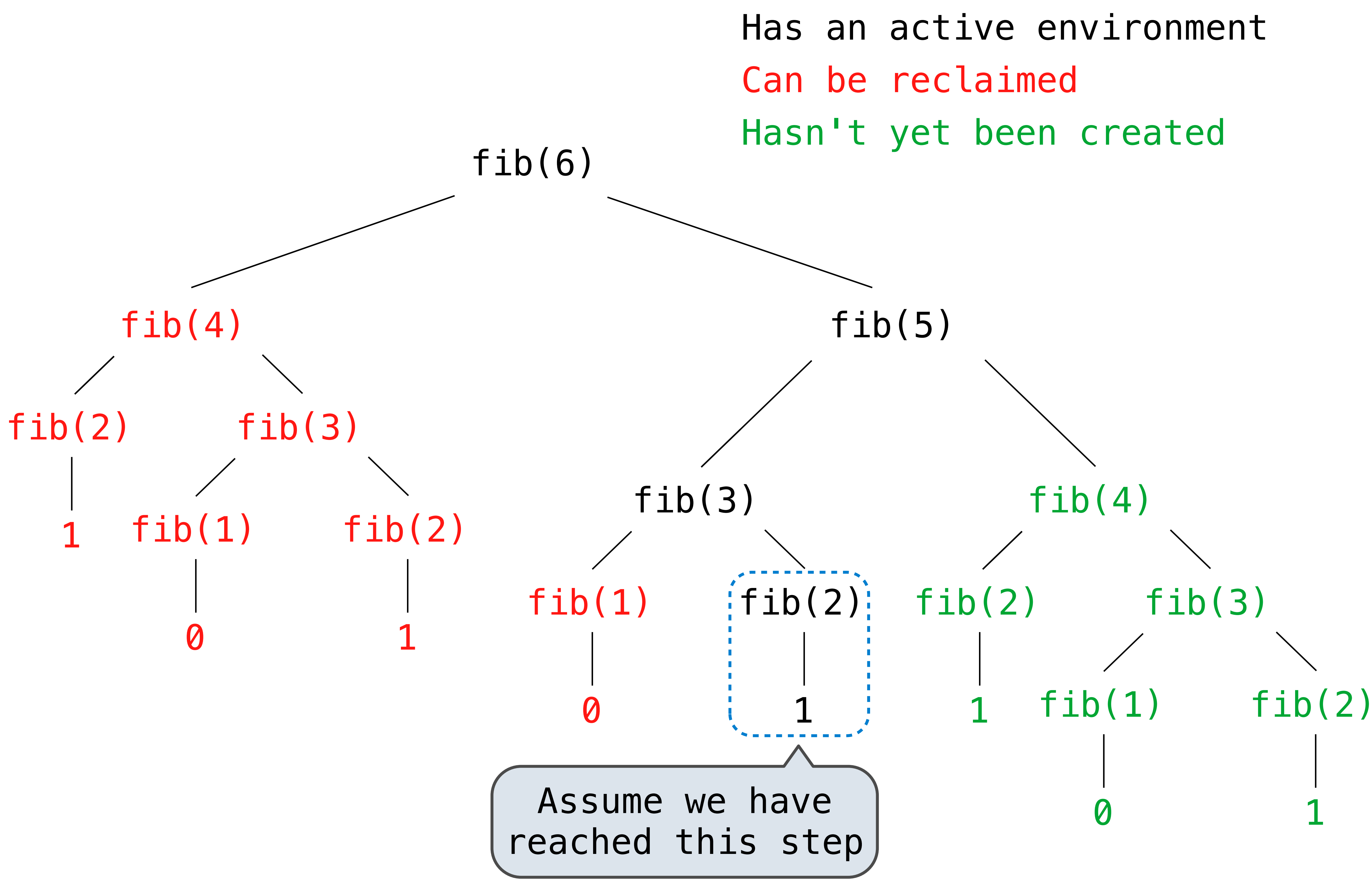
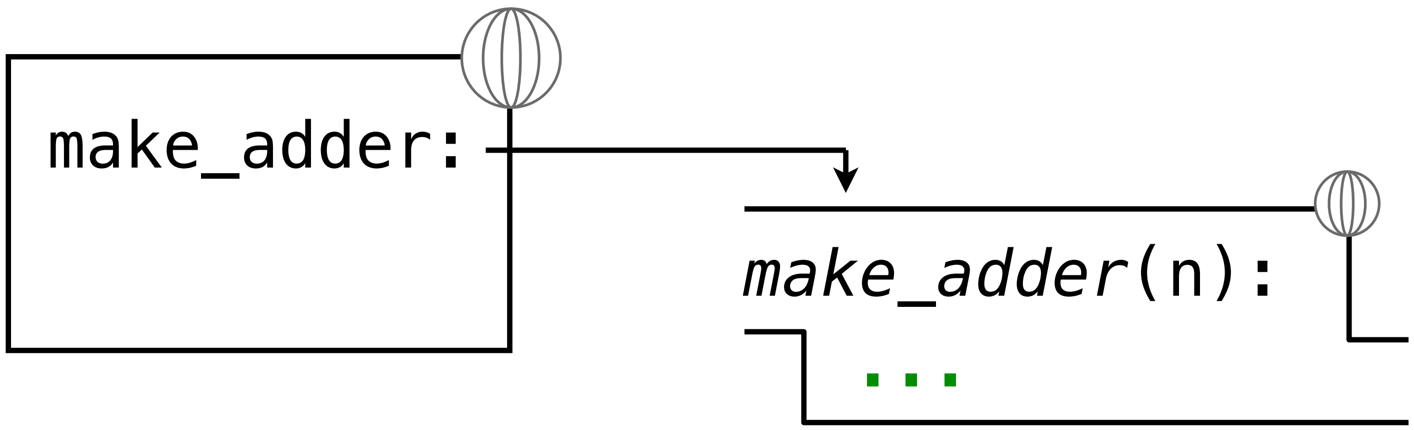


Active Environments for Returned Functions

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def make_adder(n):
    def adder(k):
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    return adder
add1 = make_adder(1)
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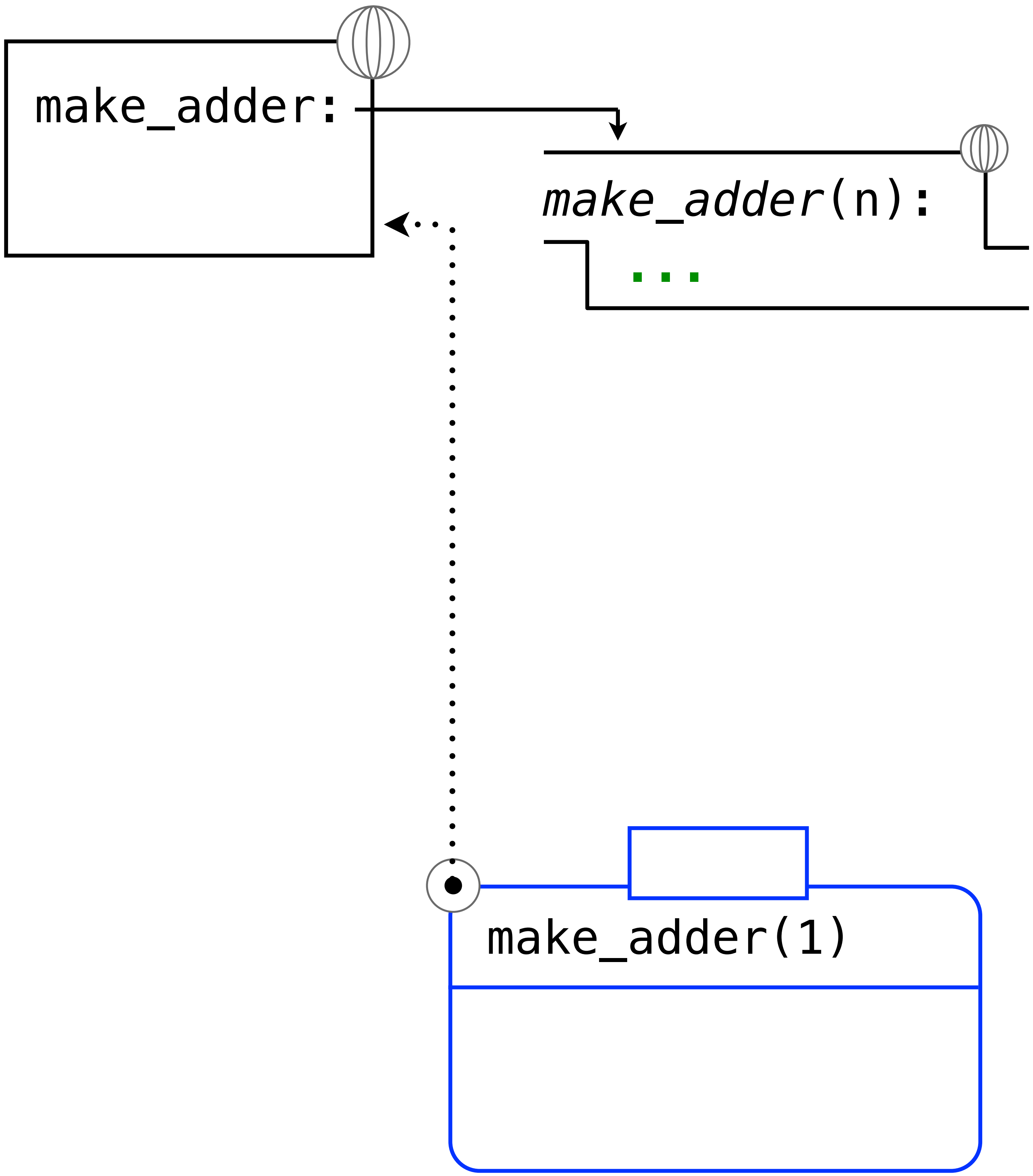


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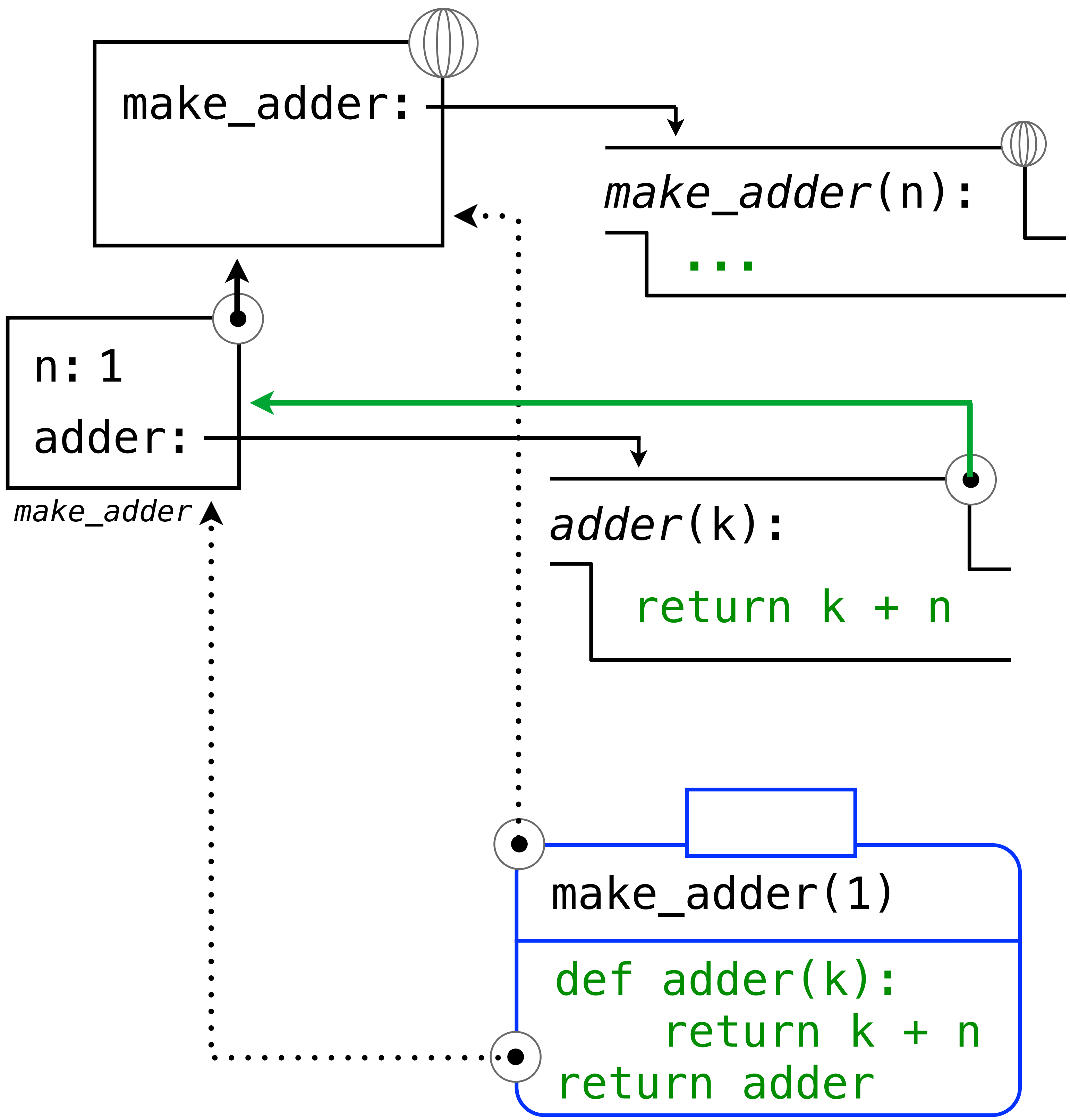


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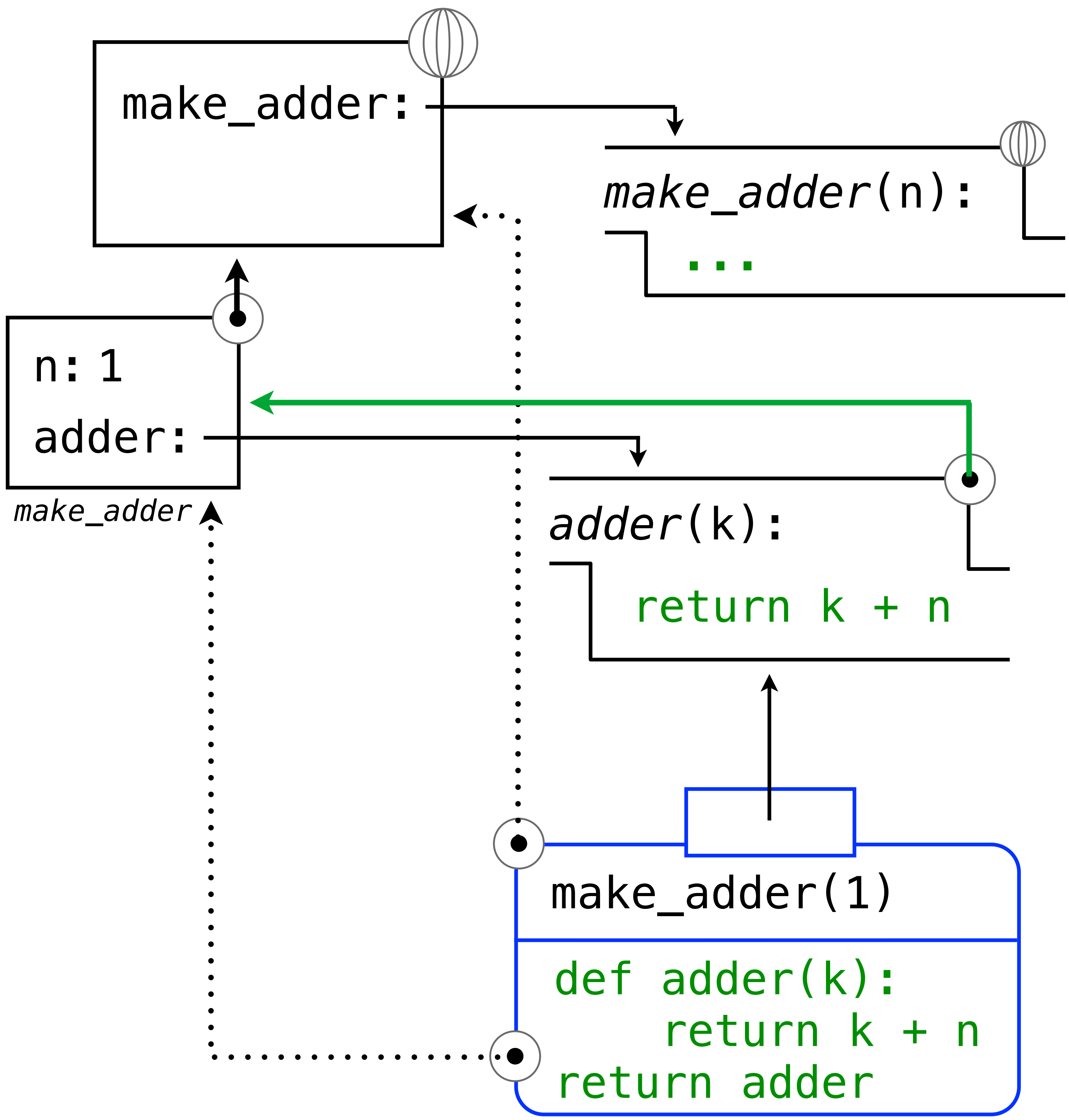


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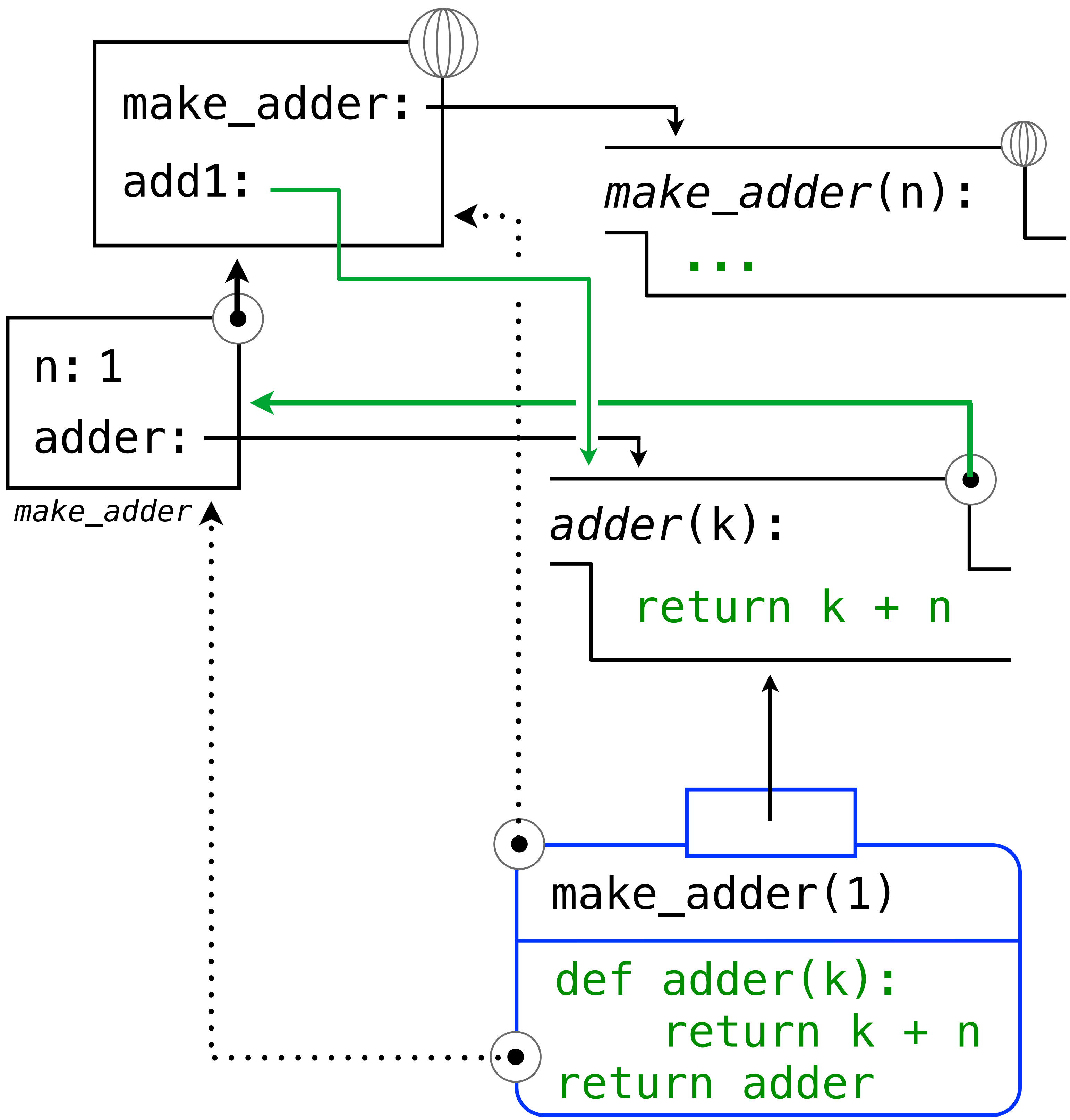


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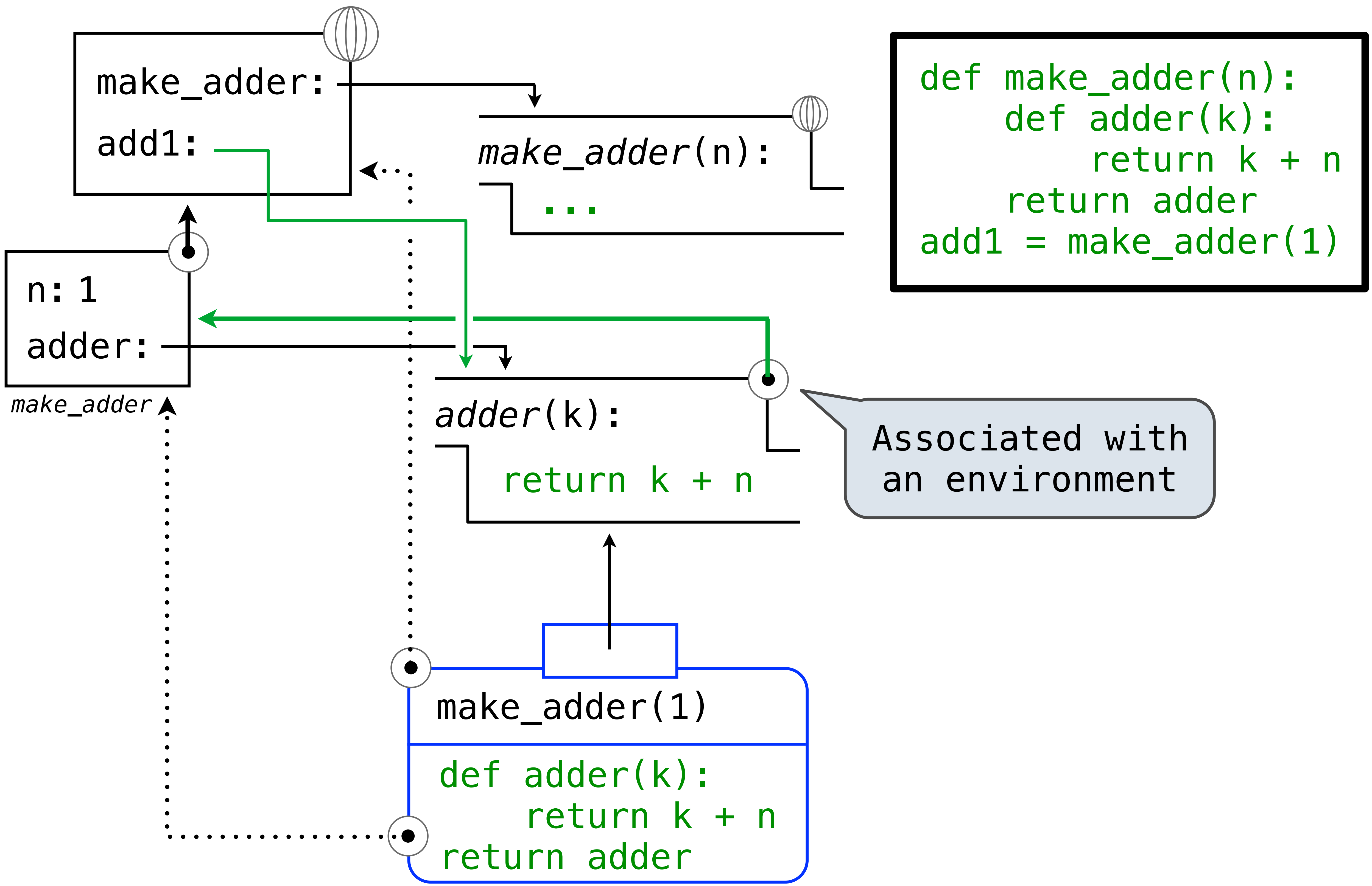

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