Reference Definitions

Inner Product Algebraic definition: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \in \mathbb{R}^N : \langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^N x_i \cdot y_i$.

Euclidean Norm The *Euclidean Norm* of a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N$ is $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$

Vector Scaling Let $c \in \mathbb{R}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N$. Recall that $c \cdot \vec{x} = \begin{bmatrix} c \cdot x_1 \\ c \cdot x_2 \\ \vdots \\ c \cdot x_N \end{bmatrix}$.

1. **Investigating Inner Products** Now follow your TA as we discover some properties of inner products.

2. **Mechanical Projection**

In $\mathbb{R}^n$, the projection of vector $\vec{b}$ onto vector $\vec{a}$ is:

$$proj_{\vec{a}}\vec{b} = \frac{\langle \vec{b}, \vec{a} \rangle}{||\vec{a}||^2}\vec{a} = \frac{\langle \vec{b}, \vec{a} \rangle}{||\vec{a}||}\frac{\vec{a}}{||\vec{a}||}$$

where $\hat{a}$ is the normalized $\vec{a}$, i.e. a unit vector with the same direction as $\vec{a}$.

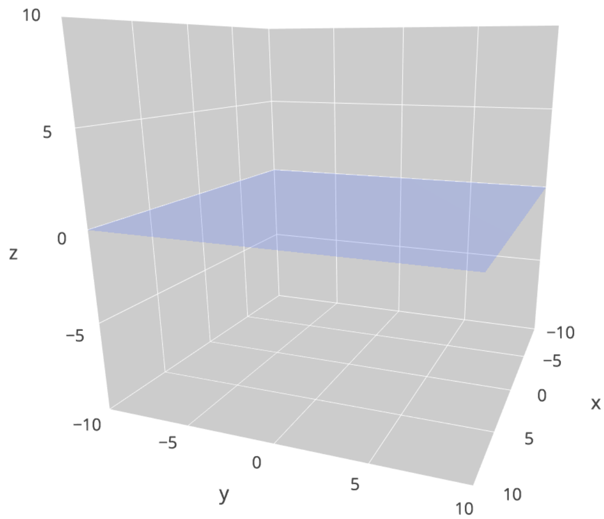
(a) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ - that is, onto the x-axis. Graph these two vectors and the projection.

(b) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ - that is, onto the y-axis. Graph these two vectors and the projection.

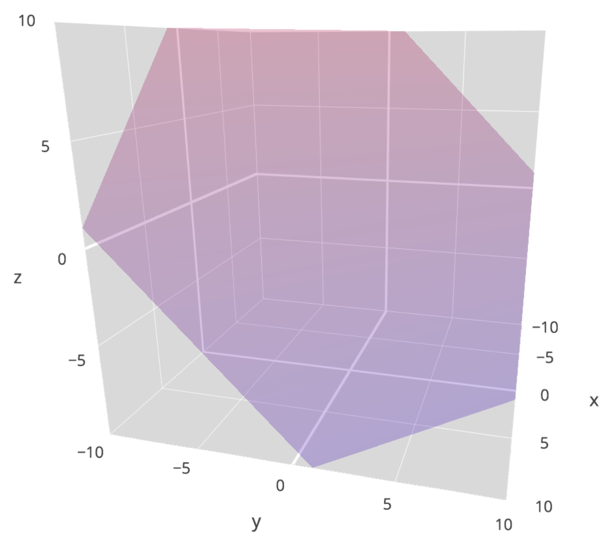
(c) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Graph these two vectors and the projection.

(d) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Graph these two vectors and the projection.

(e) Project $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the span of the vectors $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$ - that is, onto the x-y plane in $\mathbb{R}^3$. Try to visualize how this would appear on the plot below.



(f) Project $\begin{bmatrix}1\\2\\3\end{bmatrix}$ onto the plane described by $x+y+z=1$. Try to visualize how this would appear on the plot below.



(g) What is the geometric/physical interpretation of projection? Justify using the previous parts.

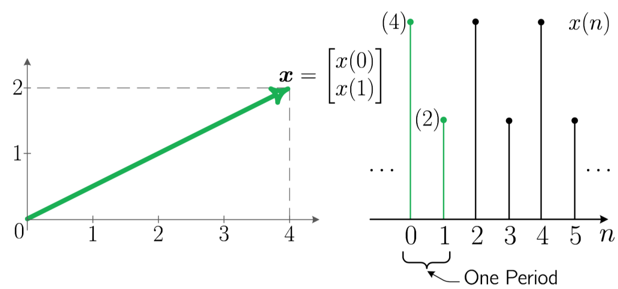
(h) For the first 4 parts, we looked at two different projections for each vector. For those cases, using only the projected vectors and the vectors we projected onto, do we have enough information to reconstruct the original vector?

(i) Given information about n projections of a vector in $\mathbb{R}^n$, when do we have enough information to reconstruct the original vector? Always? Never?

3. Periodic Signals

Periodic signals are ones that repeat themselves entirely after some time period. That is, after some time p, the signal $x(n)$ repeats itself so that $x(n+p) = x(n)$. Discrete periodic signals, during the period, do not update continuously through time. They instead update in specific discrete time steps, as if sampling a continuous signal.

Since there are a finite number of "unique" sequences in a discrete periodic signal, it is natural for us to represent the signal as a vector. We observe one period and treat the value at each time step as a different value in our vector.



Let us study the signal $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ that is periodic over $p = 2$.

(a) Write the signal as a linear combination of the standard/canonical basis. What signals do these vectors correspond to? How can we interpret the linear combination?

(b) Write the signal as a linear combination of the basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$. What signals does these vectors correspond do? How can we interpret the linear combination?

(c) Project the signal $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ onto each of the vectors in the previous part. How does these vectors relate to the linear combination from the previous part?

(d) Given the above, what is an easy way to find the coefficients for describing the signal as a linear combination of our basis? What property must hold about our basis?