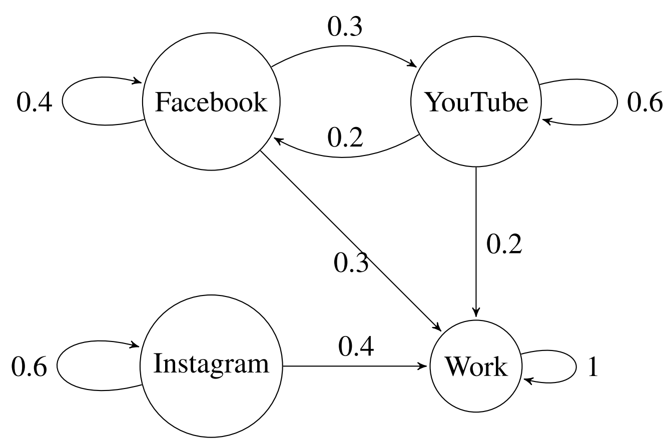
1. **Lecture Recap 9/6/16**

2. **Social Media**

   As a tech-savvy Berkeley student, the distractions of social media are always calling you away from productive stuff like homework for your classes. You're curious–are you the only one who spends hours switching between Facebook or YouTube? How do other students manage to get stuff done and balance pursuing Insta-fame? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the below figure. So, for example, if 100 students are on Facebook, in the next timestep 30 of them will click a link and move to YouTube.



   (a) What is the corresponding transition matrix?

   (b) There are 750 of you in the class. Suppose on a given Monday night (the day before HW is due), there are 350 EE16A students on Facebook, 225 on YouTube, 100 on Instagram and 75 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix one timestep, what is the state vector?

   (c) If the entries in each of the column vectors of your state transition matrix summed to 1, what would this mean with respect to the students on social media? (What is the physical interpretation?)

   (d) I want to predict how many students will be on each website n timesteps in the future. How would I formulate that mathematically? Without working it out, can you predict roughly how many students will be in each state 1000 timesteps/days in the future?

   (e) **Extra for Experts:** Suppose instead of having 'Work' as an explicit state, we assume that any student not on Facebook/Youtube/Instagram is working. Work is like the "void", and if a student is "leaked" from any of the other states we assume s/he has gone to work and will never come back. How would you reformulate this problem? Redraw the figure and rewrite the appropriate transition matrix. What are the major differences between this problem and the previous one?

3. Visualizing Matrices as Operators This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix", we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix", we will see it be "reflected". The way we will see this is by applying the operation to all the vertices of a polygon, and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled or reflected using matrices!

Part 1: Rotation Matrices as Rotations

(a) We are given matrices T_1 and T_2, and we are told that they will rotate the unit square by 15 degrees and 30 degrees, respectively. Design a procedure to rotate the unit square by 45 degrees using only T_1 and T_2, and plot the result in the iPython notebook. How would you rotate the square by 60 degrees?

(b) Try to rotate the unit square by 60 degrees using only one matrix. What does this matrix look like?

(c) T_1, T_2, and the matrix you used in part c) are called "rotation matrices". They rotate any vector by an angle, θ. Show that a rotation matrix has the following form:

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1}$$

where θ is the angle of rotation. (Hint: Use your trigonometric identities!)

(d) Now, we want to get back the original unit square from the rotated square in part b). What matrix should we use to do this? *Don't use inverses!*

(e) Use part d) to obtain the "inverse" rotation matrix for a matrix that rotates a vector by θ. Multiply the inverse rotation matrix with the rotation matrix, and vice-versa. What do you get?

Part 2: Commutativity of Operators A natural next question to ask is the following: Does the *order* in which you apply these operations matter? Follow your TA to obtain the answers to the following questions!

(a) Let's see what happens to the unit square when we rotate the matrix by 60 degrees, and then reflect it along the y-axis.

(b) Now, let's see what happens to the unit square when we first reflect it along the y-axis, then rotate the matrix by 60 degrees.

(c) Try to do steps a) and b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

(d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?

4. Derivations & Dependence

(a) Suppose we have an experiment where we have n measurements of linear combinations of n unknowns. Show that if at least one of the experiment's measurements could be predicted from the other measurements, there would be either infinite or no solutions. In other words, prove that if a $n \times n$ matrix A has rows that are linearly dependent, there will be either infinite or no solutions to $A\vec{x} = \vec{b}$.

(b) Prove that if a matrix's columns are linearly dependent, there will be either infinite or no solutions to $A\vec{x} = \vec{b}$. What is the physical interpretation of this statement?

(c) Prove that if a matrix is in $\mathbb{R}^{m \times n}$, $n \geq m$, it can have at most m linearly independent column vectors. *Hint*: Use the transpose.

(d) **Extra for Experts:** Prove that for a $m \times n$ matrix, the number of linearly independent vectors (both column and row) is at most $\min(m, n)$.