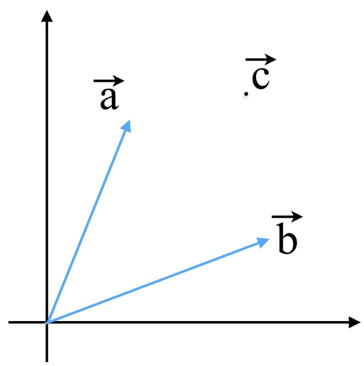
1. Lecture Recap 9/8/16

2. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β such that we reach point $\vec{c}$. That is, $\alpha\vec{a}+\beta\vec{b}=\vec{c}$.



(a) First, consider the case where $\vec{a}=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{b}=\begin{bmatrix}1\\1\end{bmatrix}$, and $\vec{c}=\begin{bmatrix}-2\\2\end{bmatrix}$. Find the two scalars α and β such that we reach point $\vec{c}$. What if $\vec{a}=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b}=\begin{bmatrix}2\\1\end{bmatrix}$?

First set: $\alpha=-4, \beta=2$, Second set: $\alpha=6, \beta=-4$

(b) Now formulate the general problem as a system of linear equations.

$$\alpha a_1+\beta b_1=c_1 \tag{1}$$
$$\alpha a_2+\beta b_2=c_2 \tag{2}$$

(c) Write this in matrix form.

$$\begin{bmatrix}a_1 & b_1\\a_2 & b_2\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}=\begin{bmatrix}c_1\\c_2\end{bmatrix} \tag{3}$$

3. Mechanical Problems

For the following problems determine whether $\vec{b}$ lies in the range of A.

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix}$
yes

(b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$
no

(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$
no

(d) $A = \begin{bmatrix} 6 & 9 & 12 \\ 9 & 5 & 7 \\ 12 & 8 & 9 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ 8 \\ 9 \end{bmatrix}$
yes

(e) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix}$
yes

(f) $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$
no

4. Span Proofs

Given some set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$, show the following

(a) The $span\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} = span(\{\alpha\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\})$, where α is a non-zero scalar. In other words, we can scale our spanning vectors and not change the span.

(b) The $span(\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}) = span(\{\vec{v}_2, \vec{v}_1, ..., \vec{v}_n\})$. In other words, we can swap the order of our spanning vectors and not change the span.

(c) The $span(\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}) = span(\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, ..., \vec{v}_n\})$. In other words, we can add our spanning vectors to one another and not change the span.

(a) Suppose $\vec{q} \in span\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_2\vec{v}_n = (\frac{a_1}{\alpha})\alpha\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_2\vec{v}_n$$

Scalar multiplication cancels out. Thus, the spans are the same

(b) Suppose $\vec{q} \in span\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_2\vec{v}_n = a_2\vec{v}_2 + a_1\vec{v}_1 + \cdots + a_2\vec{v}_n$$

Swapping the order in addition does not affect the sum, so the spanned spaces are the same.

(c) Suppose $\vec{q} \in span\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_2\vec{v}_n = a_1(\vec{v}_1 + \vec{v}_2) + (-a_1 + a_2)\vec{v}_2 + \cdots + a_2\vec{v}_n$$

We can change the scalar values to adjust for the combined vectors. Thus, the spans are the same.

5. **Inverses**

In general, an *inverse* of a matrix "undoes" the operation that that matrix performs. Mathematically, we write this as

$$A^{-1}A = I \tag{4}$$

where A^{-1} is the inverse of A. Intuitively, this means that applying a matrix to a vector and then subsequently applying it's inverse is the same as leaving the vector untouched.

Properties of Inverses. For a matrix A, if its inverse exist, then:

$$A^{-1}A = AA^{-1} = I \tag{5}$$
$$(A^{-1})^{-1} = A \tag{6}$$
$$(kA)^{-1} = k^{-1}A^{-1} \qquad \text{for a nonzero scalar } k \tag{7}$$
$$(A^T)^{-1} = (A^{-1})^T \tag{8}$$
$$(AB)^{-1} = B^{-1}A^{-1} \qquad \text{assuming } A, B \text{ are both invertible} \tag{9}$$

(a) Prove that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$$\begin{aligned} C^{-1}B^{-1}A^{-1}(ABC) &= C^{-1}B^{-1}IBC && (10)\\ &= C^{-1}IC && (11)\\ &= I && (12) \end{aligned}$$

Now consider the three matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad D = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

(b) What do each of these matrices do when you multiply them by a vector $\vec{x}$? Draw a picture.

(c) Intuitively, can these operations be undone? Why or why not? Make an intuitive argument.

(d) Are the matrices A, B, C, D invertible?

(e) Can you find anything in common about the rows (and columns) of A, B, C, D? (Bonus: How does this relate to the invertibility of A, B, C, D?)

(f) Are all square matrices invertible?

(g) How can you find the inverse of a general $n \times n$ matrix?

(b)
- A: projection on the x-axis
- B: projection on the y-axis
- C: projection on the vector $[1,\ 1]^T$
- D: projection on the vector $[1,\ 2]^T$

(c) Intuitively, none of these operations can be undone because any two vectors that lie on a line orthogonal to the axis of projection get projected to the same vector. (Draw a picture to see this, or demo this.) Applying these transformations causes loss of information. Thus, if you try to reverse the operation (taking the inverse), you can't determine which vector you started with.

(d) Since the operations are not one-to-one reversible, $A,\ B,\ C,\ D$ are not invertible.

(e) The rows of A, B, C, D are all linearly dependent with each other. The same is true for the columns. The generalization is that if a matrix is not invertible, then its rows and columns will be linearly dependent.

(f) No. We have seen in the above parts that there are square matrices that are not invertible

(g) We know that $A(A^{-1}) = I$. If we treat this as our now familiar $Ax = b$, we can use Gaussian elimination in the form:

$$\begin{bmatrix} A | I \end{bmatrix} \implies \begin{bmatrix} I | A^{-1} \end{bmatrix}$$

6. Challenging Problem

It turns out that for every set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$, there exists another set of vectors $\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_n\}$ such that $span(\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}) = span(\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_n\})$, and for each spanning vector $\vec{u}$ (that is nonzero) there is a row for which it has the element one and the other vectors have zero. We will show this next.

Recall the span proofs from part 3. Notice that the operations exactly correspond to the row operations you did while performing Gaussian elimination. We can, in fact, use these operations to better visualize the span of n vectors! Consider a set of n row vectors:

$$V^T = \begin{bmatrix} \vec{v}_1{}^T \\ \vec{v}_2{}^T \\ \vdots \\ \vec{v}_n{}^T \end{bmatrix} \tag{13}$$

The span of the rows of this matrix is precisely $span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\})$. Now, apply the same row operations as you did for Gaussian elimination to this matrix. From the properties of span, you will not be changing the span of these rows, and therefore, you will not be changing $span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\})$.

Completing Gaussian elimination gives us a matrix in reduced row-echelon (upper triangular) form comprised of row vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$. Since row operations preserve the span of the rows, we will have

$$span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}) = span(\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}) \tag{14}$$

As an example, suppose we compose our V^T matrix of the vectors $\{ \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix} \}$:

$$V^T = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 0 & 4 \\ 6 & 4 & 10 \\ -2 & 4 & 2 \end{bmatrix}. \tag{15}$$

Row reducing this matrix yields

$$U^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{16}$$

We ignore the zero vectors here. Thus $span(\{ \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix} \}) = span(\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \})$.

Question: Assuming you did not know that the second set of vectors were obtained by applying row operations to the first set of vectors, how would you prove that they had the same span? Hint: Do the second set of vectors lie in the span of the first set of vectors and vice versa?

The definition for span is:

$$span\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \{\vec{q} \in \mathbb{R}^n | \vec{q} = \sum_{i=1}^{n} \alpha_i \vec{v}_i\}$$

for some scalars α_i. We can show that each of the vectors in our V set (the original 4 vectors) can be written as a linear combination of the vectors in our U set (the latter 2 vectors).

$$\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, any vector in that is a linear combination of the V vectors can be written as a linear combination of the U vectors. By our definition of span, these sets span the same space.

When solving a system of equations, there often is no solution, as we have discovered. Practically, you then ask what is the "closest" solution in the span of our matrix A. We will reinvestigate this later.