Reference Definitions

Vector spaces: A vector space V is a set of elements that is closed under vector addition and scalar multiplication. For V to be a vector space, the following conditions must hold for every $\vec{u}, \vec{v}, \vec{z} \in V$ and for every $c, d \in \mathbb{R}$

No escape property (addition) $\vec{u} + \vec{v} \in V$,

No escape property (scalar multiplication) $c\vec{u} \in V$,

Commutativity $\vec{u} + \vec{v} = \vec{v} + \vec{u}$,

Associativity of vector addition $(\vec{u} + \vec{v}) + \vec{z} = \vec{u} + (\vec{v} + \vec{z})$,

Additive identity There is $\vec{0} \in V$ such that for all $\vec{u}$, $\vec{0} + \vec{u} = \vec{u} + \vec{0} = \vec{u}$,

Existence of inverse For every $\vec{u}$, there is element $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = 0$,

Associativity of scalar multiplication $c(d(\vec{u})) = (cd)\vec{u}$,

Distributivity of scalar sums $(c + d)\vec{u} = c\vec{u} + d\vec{u}$,

Distributivity of vector sums $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$,

Scalar multiplication identity There is $1\vec{u} = \vec{u}$.

The most important of the above properties are the *no escape properties*. These demonstrate that the vector space is closed under addition and scalar multiplication. That is, if you add two vectors in V, your resulting vector will still be in V. If you multiply a vector in V by a scalar, your resulting vector will still be in V.

Subspaces: A subset W of a *vector space* V is a *subspace* of V if the following two conditions hold for any two vectors $\vec{u}, \vec{v} \in W$, and any scalar $c \in \mathbb{R}$:

No escape property (addition) $\vec{u} + \vec{v} \in W$

No escape property (scalar multiplication) $c\vec{u} \in W$

Note that these are the only properties we need to establish to show that a subset of a vector space is a subspace! The other properties of the underlying vector space come for free, so to speak.

The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$, as well as their subspaces.

Basis: A *basis* for a vector space is a *set of linearly independent vectors* that *spans the vector space*.

So, if we want to check whether a set of vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ forms a basis for a vector space V, we check for two important properties:

(a) $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ are linearly independent.

(b) $span(\vec{v}_1, \vec{v}_2, ..., \vec{v}_k) = V$

As we move along, we'll learn how to identify and/or construct a basis, and we'll also learn some interesting properties of bases.

1. Lecture Review 9/13/16

2. Identifying a subspace: Proof exercise!

Is the set

$$V = \left\{ \vec{v} : c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?

Yes, V is a subspace of $\mathbb{R}^3$. We will *prove this* by using the definition of a subspace.

First of all, note that V is a subset of $\mathbb{R}^3$ – any element in V looks like $\begin{bmatrix} c+d \\ c \\ c+d \end{bmatrix}$, which is clearly a 3-dimensional real vector.

Now, consider two elements $\vec{v}_1, \vec{v}_2 \in V$, and $\alpha \in \mathbb{R}$.

This means that there exists $c_1, d_1 \in \mathbb{R}$ such that $\vec{v}_1 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Similarly, there exists $c_2, d_2 \in \mathbb{R}$ such that $\vec{v}_2 = c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Now, it is clear that

$$\vec{v}_1 + \vec{v}_2 s = (c_1 + c_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (d_1 + d_2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

and therefore, $\vec{v}_1 + \vec{v}_2 \in V$.

Also,

$$\alpha \vec{v}_1 = (\alpha c_1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (\alpha d_1) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

and therefore, $\alpha\vec{v_1} \in V$.

We have shown both the no escape properties, and so V is a subspace of $\mathbb{R}^3$.

3. Identifying a basis

Does each of these sets describe a basis of some vector space?

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- Yes, the vectors are linearly independent and so they are a basis of some 2-dimensional subspace.
- Yes, the vectors are linearly independent and will form a basis of $\mathbb{R}^3$.
- No, $\vec{v_3}, 2 + \vec{v_3}, 3 = \vec{v_3}, 1$ and so the vectors are not linearly independent.

4. Constructing a basis

Let's consider a subspace of $\mathbb{R}^3$, V, that has the following property: for every vector in V, the first entry is equal to two times the sum of the second and third entries. That is, if $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V$, we have $a_1 = 2(a_2 + a_3)$.

Find a basis for V. What is the dimension of V?

Note that any vector $\vec{v}$ in V is going to look as follows:

$$\vec{v} = \begin{bmatrix} 2(a_2 + a_3) \\ a_2 \\ a_3 \end{bmatrix} = a_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

Now, we consider the set of vectors

$$B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

. The vectors are linearly independent. Furthermore, from the above equation, any vector $\vec{v} \in V$ can be expressed as a linear combination of the vectors in B! (corresponding coefficients are a_2 and a_3.) This means that $V = span(B)$.

Therefore

$$B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

forms a basis for V.

Clearly, $|B| = 2$, and so the dimension of V is 2.

5. Exploring dimensionality, linear independence and bases

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimensionality of a vector space/subspace, and basis.

Let's consider the vector space $\mathbb{R}^m$, and a set of n vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ in $\mathbb{R}^m$.

(a) For the first part of the problem, let $m > n$. Can $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ form a basis of $\mathbb{R}^m$? Why/why not? What conditions would we need?

No, clearly $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ cannot form a basis of $\mathbb{R}^m$. The dimension of $\mathbb{R}^m$ is m, so you would need m linearly independent vectors to describe the vector space. Since $n < m$, this is not possible.

(b) Let $m = n$. Can $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ form a basis of $\mathbb{R}^m$? Why/why not? What conditions would we need?

Yes, this is possible. The only condition we need is that $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ are all linearly independent. If the vectors are linearly independent, since there are m of them, they will span $\mathbb{R}^m$.

(c) Now, let $m < n$. Can $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ form a basis of $\mathbb{R}^m$? What vector space could they form a basis for? (Hint: think about whether the vectors can now be linearly independent.)

Clearly, $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ cannot form a basis of $\mathbb{R}^m$ because the dimension of the vector space is m, and we have n vectors. This is easy to see. Since $n > m$, some of the vectors have to be linearly dependent, and so, they cannot form a basis.

The two regimes – one in which $n > m$ – and one in which $n < m$ – give rise to two different classes of interesting problems. You might learn more about them in upper division courses!

6. Are Some Bases Better Than Others?

In general there can be many bases for the same vector space. To see this, let's consider the vector space $\mathbb{R}^3$. Clearly,

$$V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis of $\mathbb{R}^3$. This is called the *standard basis*.

(a) Show that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is also a basis of $\mathbb{R}^3$.

The dimension of the space is 3 (it is $\mathbb{R}^3$), and there are 3 linearly independent vectors. They must span the whole space.

(b) Show that $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is also a basis of $\mathbb{R}^3$.

The dimension of the space is 3 (it is $\mathbb{R}^3$), and there are 3 linearly independent vectors. They must span the whole space.

(c) Which of the two bases might you prefer to use to describe $\mathbb{R}^3$? Why?

The second basis is an orthonormal basis. We are going to see towards the end of this course that orthonormal bases have very nice properties.