EECS 16A Designing Information Devices and Systems I
Fall 2016 Official Lecture Notes **Note 5**

Water Reservoirs and Pumps Examples

One way of visualizing matrix-vector multiplication is by considering water reservoirs and water pumps. It is vital as an engineer that you understand the ideas that we are talking about in an intuitive way. This will often come from having a series of examples that make sense. After all, we define mathematical operations the way that we do because these definitions are useful. They don't come out of nowhere. The act of doing mathematics and engineering is often about making definitions and seeing where they lead us, while checking the consistency of these definitions with what we are trying to model in the real world.

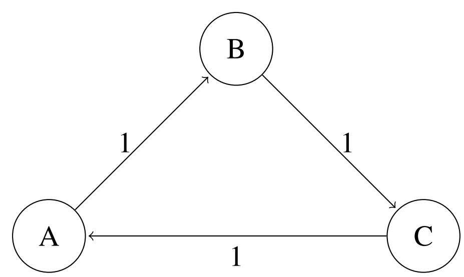
For these examples, we will have three water reservoirs, A,B,C. Let's say the initial amounts of water they respectively hold are A_0,B_0,C_0. Now we have a system of pumps connecting the reservoirs to move certain amounts of water between the reservoirs.

We can represent the reservoirs as a vector $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ where each element is how much water is currently in that reservoir. Then, we can represent the system of pumps as a matrix: $\begin{bmatrix} A_a & B_a & C_a \\ A_b & B_b & C_b \\ A_c & B_c & C_c \end{bmatrix}$.

Each element K_i represents the fraction of reservoir K that goes into reservoir i. The matrix acts on the vector just as the pumps acts on the reservoirs, performing a transformation. This example can also extend to matrix-matrix multiplication. In addition to this, this and a similar example (PageRank — how search engines can use link information to figure out which pages are important) will show further applications of linear algebra.

Basic Pump

The most basic pump system will move all of the water from one pump into another. Pictorially, the blue circles are the reservoirs and the arrows represent how the pumps move the water:



The corresponding matrix-vector multiplication is:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

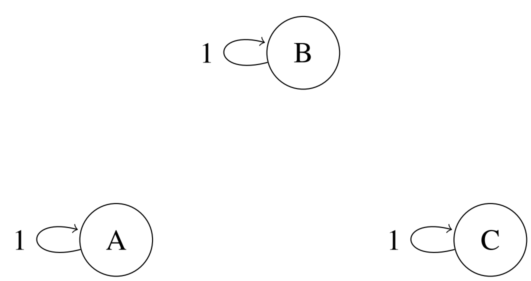
Each time the pumps act on the reservoirs, all of the water in reservoir A flows into reservoir B. All of the water in reservoir B flows into reservoir C. All of the water in reservoir C flows into reservoir A. If A, B, and C all start with the same amount of water, then the pumps acting on the reservoirs would not change the amount of water in each reservoir. As an example, let the amount of water in each reservoir initially by A_0, B_0, C_0. Then calculating the amount of water in each reservoir after activating the pumps once can be done this way:

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} C_0 \\ A_0 \\ B_0 \end{bmatrix}$$

Identity Matrix Pump

What happens when your pump system can be represented as the identity matrix? What does that mean? Well if the initial amounts of water in the reservoirs are represented by the vector $\begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix}$, and the identity matrix represents how the pumps move the water, after one activation of the pumps, nothing changes!

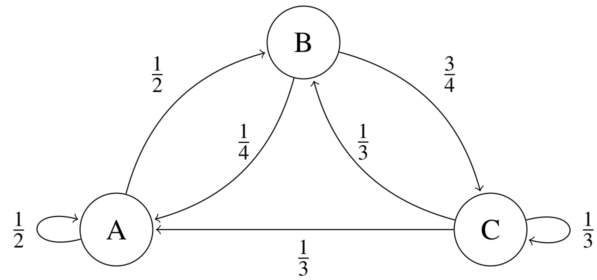
$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix}$$



Conservation of Water

Now let's look at what happens when pumps move different amounts of water from each reservoir into other reservoirs. Specifically, let's work with this diagram: Let us describe these pumps with this matrix:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix}$$



Each element of the matrix still represents a pump and indicates how much water is moved where. The first row indicates how much of each reservoir contributes to reservoir A when the pumps are activated. The second row does the same for reservoir B, and the third row is for reservoir C. For example, the upper left element $\frac{1}{2}$ tells us that half of what is in reservoir A will stay in reservoir A. Similarly, the $\frac{1}{2}$ on the middle left tells us that the other half of what was in reservoir A will flow into reservoir B when the pumps turn on. As we can see, each column of the matrix sums to one. This means the water is conserved (none mysteriously lost or gained). The water will either stay in the original reservoir or move to a different one.

This is a useful fact about water moving between pools with no evaporation, but it is not something that is going to hold in all useful applications of matrices.

After activating the pumps once, how do we know how much water is in each reservoir? That is calculated with a simple matrix-vector multiplication:

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot A_0 + \frac{1}{4} \cdot B_0 + \frac{1}{3} \cdot C_0 \\ \frac{1}{2} \cdot A_0 + 0 \cdot B_0 + \frac{1}{3} \cdot C_0 \\ 0 \cdot A_0 + \frac{3}{4} \cdot B_0 + \frac{1}{3} \cdot C_0 \end{bmatrix}$$

Drain

Another special matrix is the zero matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In terms of the reservoirs, it would be some sort of drain (e.g. an evil monster evaporated all of the water in the three reservoirs). The zero matrix acting on the reservoirs results in zero water left in each reservoir.

This doesn't obey water conservation but it is still representable as a matrix.

Special Matrices

A **zero matrix** is a matrix with all the components equal to zero, usually just represented as 0, where its size is implied from context.

An **identity matrix** is a square matrix whose diagonal elements are 1 and whose off-diagonal elements are all 0:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the column vectors (and the transpose of the row vectors) of an $n \times n$ identity matrix are the unit vectors in $\mathbb{R}^n$.

The **transpose** of an $m \times n$ matrix A, denoted A^T is the $n \times m$ matrix given by $(A^T)_{ij} = A_{ji}$.

A square matrix is said to be **symmetric** if $A = A^T$, which means that $A_{ij} = A_{ji}$ for all i and j.

Column vs Row Perspective

This pumps and pools example makes it clear that both the rows of a matrix and the columns of a matrix are interpretable.

The rows represent the perspective of the next time's pool — each row tells how much proportion of water it will draw from the particular pools at this time.

The columns represent the perspective of this time's water in a particular pool — each column tells that water in what proportion will it end up in the other pools at the next time.

This complements the interpretation of rows and columns that come from the system-of-linear-equations perspective of doing experiments. There, each row of the experiment matrix represented a particular experiment that took one particular measurement. The coefficients represented how much that particular state variable effects the outcome of this particular experiment.

By contrast, in the experiment-perspective, the columns represented the influence of a particular state variable on the collection of experiments taken together.

These perspectives come in handy in interpreting matrix multiplication.

Matrix Addition

Two matrices of the same size can be added together by adding the corresponding components.

$$\begin{bmatrix} -1 & 3 \\ 3.5 & 2 \\ 0 & -0.1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & -2 \\ 3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2.5 & 0 \\ 3 & 0 \end{bmatrix}$$

Matrix addition has the following properties:

(a) **Commutativity** $A + B = B + A$

(b) **Associativity** $(A + B) + C = A + (B + C)$

(c) **Additive Identity (Zero)** $A+0=A$

(d) **Additive Inverse** $A+(-A)=0$

Scalar Matrix Multiplication

As with vectors, multiply each component of the matrix by the scalar:

$$(-3)\begin{bmatrix} -1 & 3 \\ 3.5 & 2 \\ 0 & -0.1 \end{bmatrix} = \begin{bmatrix} 3 & -9 \\ -10.5 & -6 \\ 0 & 0.3 \end{bmatrix}$$

Additionally, we know that $-A=(-1)A$, and $(0)A=0$.
For scalars α,β and matrices A,B:

(a) **Associativity** $(\alpha\beta)A=(\alpha)(\beta A)$

(b) **Distributive** $(\alpha+\beta)A=\alpha A+\beta A$ and $\alpha(A+B)=\alpha A+\alpha B$

(c) **Multiplicative Identity (One)** $(1)A=A$

Matrix-Vector Multiplication

A matrix and a vector, $A\in\mathbb{R}^{m\times n}, x\in\mathbb{R}^n$, can be multiplied together as follows:

$$Ax=\begin{bmatrix} A_{11}x_1 & A_{12}x_2 & \dots & A_{1n}x_n \\ A_{21}x_1 & A_{22}x_2 & \dots & A_{2n}x_n \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ A_{m1}x_1 & A_{m2}x_2 & \dots & A_{mn}x_n \end{bmatrix}$$

here's an example:

$$\begin{bmatrix} -1 & 3 \\ 3.5 & 2 \\ 0 & -0.1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -0.1 \end{bmatrix}$$

How can this be interpreted? The interpretation is that a matrix multiplying a vector returns a weighted sum of the columns in the matrix. Where the weights are given by the vector that is being multiplied. When the vector is a standard unit vector, this just selects a particular column of the matrix.

Notice that this makes sense in both interpretations. In the pumps and pools interpretation, multiplying by the matrix tells us where the water ends up. If the water from the first pool ends up divided as per the first column of the matrix while water from the second pool ends up divided as per the second column of the matrix, then if water starts out in a weighted combination of being in the first and second pool, it should also end up in the same combination of the two columns of the matrix!

Meanwhile, in the experiments and measurements perspective, the columns represent the influence of each variable on the results of the experiment. The final measurements represent a weighted combination of the influence of each variable, where the weights are given by the actual values that each variable has taken.

Matrix Multiplication i.e. Transformation of Spaces

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \underset{B}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}} = \underset{AB}{\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}}$$

Computationally, we can think of matrix multiplication as cascading – we take the inner product of each row vector with the column vector of the other matrix starting from the first column of matrix B and going to the right. You can think of it as the first matrix acting on each column of the second matrix to produce a new column. Why columns and not rows? That's just convention. But this does lead to an important point about the dimensions of matrix-matrix multiplication. If you want to multiply matrix A and B ($A \cdot B$), then the number of columns in matrix A must equal the number of rows in matrix B. Otherwise, $A \cdot B$ cannot be done. So if A is an $m \times n$ matrix and B is $n \times p$, the result of $A \cdot B$ will have dimensions $m \times p$.

Another description:

$$\begin{bmatrix} ----R_1---- \\ ----R_2---- \\ . \\ . \\ . \\ ----R_N---- \end{bmatrix} \begin{bmatrix} | & | & | \\ | & | & | \\ | & | & | \\ C_1 & C_2 \dots & C_N \\ | & | & | \\ | & | & | \\ | & | & | \end{bmatrix} = \begin{bmatrix} R_1C_1 & R_1C_2 & \dots & R_1C_N \\ R_2C_1 & R_2C_2 & \dots & R_2C_N \\ & . & & \\ & . & & \\ & . & & \\ R_NC_1 & R_NC_2 & \dots & R_NC_N \end{bmatrix}$$ where each matrix is an $N \times N$ matrix.

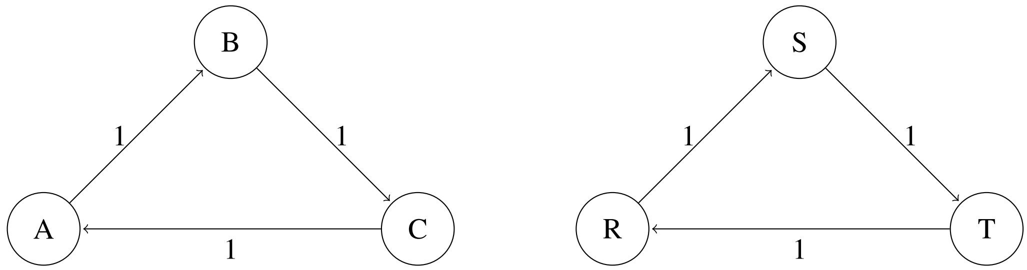
Example 5.1 (Matrix Multiplication): $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (2)(1)+(4)(3) & (2)(2)+(4)(4) \\ (3)(1)+(1)(3) & (3)(2)+(1)(4) \end{bmatrix} = \begin{bmatrix} 14 & 20 \\ 6 & 10 \end{bmatrix}$

Example of Matrix-Matrix Multiplication

Let's revisit the water reservoirs and pumps. This example can be extended from matrix-vector multiplication to matrix-matrix multiplication in different ways.

Twin Cities

In this scenario, we have two cities that each have three reservoirs. The pump systems in the cities are identical. Let's start with the basic pump system.



(a) City 1 (b) City 2

The pumps can still be represented as a matrix, like before:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

But now, instead of acting on just one set of reservoirs, the pumps act on two. So what if now, instead of the pump matrix acting on one vector representing the reservoirs, the matrix acts on two vectors? But the two vectors representing the water reservoirs in each city can be "stacked" into a matrix, where each column is a reservoir vector.

So in City 1, the reservoirs initially have water amounts A_0, B_0, C_0. In City 2, the reservoirs initially have water amounts R_0, S_0, T_0. Then, once the pumps act of the reservoirs, the amount of water in each reservoir can be found this way:

$$\begin{bmatrix} A_1 & R_1 \\ B_1 & S_1 \\ C_1 & T_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_0 & R_0 \\ B_0 & S_0 \\ C_0 & T_0 \end{bmatrix} = \begin{bmatrix} C_0 & T_0 \\ A_0 & R_0 \\ B_0 & S_0 \end{bmatrix}$$

If the cities have identical but more complicated pumps (such as the conservation pumps in the previous example), finding out how the reservoirs change is the same process. All that would be different is the "pump system" matrix. What if you have the same pump-reservoir system in k cities? To find out how the pumps act on the reservoirs, you can still use matrix-matrix multiplication. One matrix describes the pumps, while the other describes the reservoirs. There would be k columns in the reservoir matrix because each column is a vector that represents the reservoirs of a certain city.

So in this case, we see a matrix acting on another matrix, or transforming multiple vectors the same way. And because of this, we can also see why the dimensions of the matrix have certain restrictions. The number of columns in the pumps matrix must match the number of rows in the reservoir matrix. The pumps matrix acts on each column of the reservoir matrix to produce a new column for the resulting matrix that describes amount of water for that city's reservoirs.

Activate pumps once...and then once more

Now, imagine we have one system of pumps for one city with three reservoirs. How can we calculate the amount of water in each reservoir after activating the pumps twice? From matrix-vector multiplication, we know how to find the amount after one activation. If matrix A represents the pumps and $\vec{v}_0$ represents the initial reservoir vector, then $\vec{v}_1 = A \cdot \vec{v}_0$ will tell us how much water is in each reservoir after one activation. Then $\vec{v}_2 = A \cdot \vec{v}_1$ will tell us how much water is in each reservoir after the second activation.

But from the reservoirs' standpoints, how they got from $\vec{v}_0$ to $\vec{v}_2$ does not matter. For all they know, it could have been some other system of pumps (matrix B) that acted on the reservoirs initially that resulted in $\vec{v}_2$. This means that one set of pumps acting twice on the reservoirs is equivalent to *another* matrix acting on the reservoirs:

$$\begin{aligned} A(A \cdot \vec{v}_0) &= B \cdot \vec{v}_0 \\ (A \cdot A)\vec{v}_0 &= B \cdot \vec{v}_0 \\ A^2 &= B \end{aligned}$$

As an example, let's take the pump system from the conservation matrix-vector multiplication example:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix}, \vec{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let's calculate $\vec{v}_2$:

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{13}{12} \\ \frac{5}{6} \\ \frac{13}{12} \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{13}{12} \\ \frac{5}{6} \\ \frac{13}{12} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{65}{72} \\ \frac{71}{72} \end{bmatrix}$$

For comparison:

$$B = A^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{3}{8} & \frac{13}{36} \\ \frac{1}{4} & \frac{3}{8} & \frac{5}{18} \\ \frac{3}{8} & \frac{1}{4} & \frac{13}{36} \end{bmatrix}$$

$$B \cdot \vec{v}_0 = \begin{bmatrix} \frac{3}{8} & \frac{3}{8} & \frac{13}{36} \\ \frac{1}{4} & \frac{3}{8} & \frac{5}{18} \\ \frac{3}{8} & \frac{1}{4} & \frac{13}{36} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{65}{72} \\ \frac{71}{72} \end{bmatrix}$$

So we can see from this example that a matrix-matrix multiplication results in an equivalent matrix. Pump system A acting twice on the reservoirs is the same as pump system B acting once on the reservoirs.

A Multitude of Pumps

Another example of matrix-matrix multiplication with these pumps and reservoirs is when two (or more) *different* sets of pumps act sequentially on a city's reservoirs. From the previous example, we know that a matrix multiplied by another matrix is equivalent to another matrix. That principle can be applied here. So if we have Pump System A act on the reservoirs ($\vec{v}_0$) and then Pump System B act on the reservoirs, it is the same as if some other Pump System C acted on the reservoirs:

$$\begin{aligned} B \cdot (A \cdot \vec{v}_0) &= C \cdot \vec{v}_0 \\ (B \cdot A)\vec{v}_0 &= C \cdot \vec{v}_0 \\ B \cdot A &= C \end{aligned}$$